8 Sequences and Series

Sequence and series are useful because they can be used to model real-world situations. Geometric series occur in real life when each actor in a system behaves independently and is fixed. If each person decides not to have another child based on the current population, the annual population increase is geometric.

Topic Notes

- *Arithmetic Mean and Geometric Progression*
- *Relation Between A.M. and G.M.*

ARITHMETIC MEAN AND GEOMETRIC PROGRESSION 1

TOPIC 1

INTRODUCTION

Sequence and Series are two mathematical words that are generally considered implying the same thing, a set of numbers arranged in some definite order.

Sequences

A sequence is a function whose domain is as the subset of the set of natural numbers N and range is as the subset of real number R. A sequence is an ordered list of numbers.

Important

- *The numbers in the list are called terms.*
- *a_n is called n^{th} term of sequence $a_1, a_2, a_3, \ldots$*
- *The n^{th} term is also called the general term of the sequence*

A sequence is said to be a progression if the term of the sequence can be expressed by some explicit formula.

e.g. 2, 6, 10, 14 ..... is a sequence.

Here, $a_1 = 2, a_2 = 6, a_3 = 10$ and $a_4 = 14$.

Example 1.1: **Write first five terms of sequence.**

(A) $a_n = (-1)^{n-1}5^{n+1}$

(B) $a_n = 2n^2 - n + 1$ **[NCERT]**

Ans. (A) We have, $a_n = (-1)^{n-1}\, 5^{n+1}$

On putting $n = 1$, we get

$$a_1 = (-1)^{1-1}\, 5^{1+1}$$
$$= (-1)^0\, 5^2 = 25$$

On putting $n = 2$, we get

$$a_2 = (-1)^{2-1}\, 5^{2+1}$$
$$= (-1)^2\, 5^3 = -125$$

On putting $n = 3$, we get

$$a_3 = (-1)^{3-1}\, 5^{3+1}$$
$$= (-1)^2\, 5^4 = 625$$

On putting $n = 4$, we get

$$a_4 = (-1)^{4-1}\, 5^{4+1}$$
$$= (-1)^3\, 5^5 = -3125$$

On putting $n = 5$, we get

$$a_5 = (-1)^{5-1}\, 5^{5+1}$$
$$= (-1)^4\, 5^6 = 15625$$

Hence, the first five terms of the given sequence are 25, −125, 625, −3125, 15625.

(B) We have, $a_n = 2n^2 - n + 1$

On putting $n = 1$, we get

$$a_1 = 2(1)^2 - 1 + 1$$
$$= 2 - 1 + 1 = 2$$

On putting $n = 2$, we get

$$a_2 = 2(2)^2 - 2 + 1$$
$$= 8 - 2 + 1 = 7$$

On putting $n = 3$, we get

$$a_3 = 2(3)^2 - 3 + 1$$
$$= 18 - 3 + 1 = 16$$

On putting $n = 4$, we get

$$a_4 = 2(4)^2 - 4 + 1$$
$$= 32 - 4 + 1 = 29$$

On putting $n = 5$, we get

$$a_5 = 2(5)^2 - 5 + 1$$
$$= 50 - 5 + 1 = 46$$

Hence, the first five terms of the given sequence are 2, 7, 16, 29 and 46.

Series

If $a_1, a_2, a_3 \ldots$ be a sequence, then the expression $a_1 + a_2 + a_3 + \ldots$ is called the series associated with the given sequence.

Series can be represented in compact form, called sigma notation. It is denoted by greek letter 'Σ' (sigma) e.g. series $a_1 + a_2 + \ldots + a_n + \ldots$ can be represented in compact form as $\sum_{k=1}^{\infty} a_k$.

Example 1.2: **Let the sequence a_n is defined as follows $a_1 = 2, a_n = a_{n-1} + 3$ for $n \geq 2$.**

Find first five terms and write corresponding series.

Ans. We have, $a_1 = 2$ and $a_n = a_{n-1} + 3$

On putting $n = 2$, we get $a_2 = a_1 + 3 = 2 + 3 = 5$

On putting $n = 3$, we get $a_3 = a_2 + 3 = 5 + 3 = 8$

On putting $n = 4$, we get $a_4 = a_3 + 3 = 8 + 3 = 11$

On putting $n = 5$, we get $a_5 = a_4 + 3 = 11 + 3 = 14$

Thus, first five terms of given sequence are 2, 5, 8, 11 and 14. Also, the corresponding series is $2 + 5 + 8 + 11 + 14 + \ldots$.

TOPIC 2

ARITHMETIC MEAN

(1) If a, A and b are in AP, then $A = \frac{(a+b)}{2}$ is called the arithmetic mean of a and b.

(2) If a_1, a_2, a_3, a_n are n numbers, then their AM is given by,

$$d = \frac{b-a}{n+1}$$

$$A_1 = a + d = \frac{na+b}{n+1}$$

$$A_2 = a + 2d = \frac{(n-1)a+2b}{n+1}$$

$$\ldots A_n = a + nd = \frac{a+nb}{n+1}$$

(3) If $a, A_1, A_2, A_3, \ldots, A_n, b$ are in AP, then $A_1, A_2, A_3, \ldots, A_n$ are n arithmetic mean between a and b, where

$$d = \frac{b-a}{n+1}$$

$$A_1 = a + d = \frac{na+b}{n+1}$$

$$A_2 = a + 2d = \frac{(n-1)a+2b}{n+1}$$

$$\ldots A_n = a + nd = \frac{a+nb}{n+1}$$

(4) Sum of n AM's between a and b is nA

i.e., $A_1 + A_2 + A_3 + \ldots + = nA$

Important Results on AP

(i) If $a_p = q$ and $a_q = p$, then $a_{p+q} = 0$, $T_r = p + q - r$

(ii) If $pT_p = qT_q$, then $a_{p+q} = 0$

(iii) If $a_p = \frac{1}{p}$ and $a_q = \frac{1}{p}$, then $a_{pq} = 1$

(iv) If $S_p = q$ and $S_q = p$, then $S_{p+q} = -p + q$

(v) If $S_p = S_q$, then $S_{p+q} = 0$

Example 1.3: Show that the sequence whose n^{th} term is $a_n = n(n + 2)$ is not an A.P. [NCERT]

Ans. Given $a_n = n(n + 2) = n^2 + 2n$

Then,
$$a_{n+1} = (n + 1)^2 + 2(n + 1)$$
$$= (n^2 + 2n + 1) + (2n + 2)$$
$$= n^2 + 4n + 3$$

Now, $a_{n+1} - a_n = (n^2 + 4n + 3) - (n^2 + 2n)$
$$= 2n + 3$$

which is dependent on n.

Hence, the given sequence is not an A.P.

Example 1.4: If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is A.M. of a and b. Then find the value of n. [NCERT]

Ans. Given that $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. of a and b.

$$\therefore \quad \frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \frac{a+b}{2}$$

$$\Rightarrow \quad 2a^n + 2b^n = (a + b)(a^{n-1} + b^{n-1})$$

$$\Rightarrow \quad 2a^n + 2b^n = a^n + ab^{n-1} + ba^{n-1} + b^n$$

$$\Rightarrow \quad a^n + b^n = ab^{n-1} + ba^{n-1}$$

$$\Rightarrow \quad a^n - ba^{n-1} = ab^{n-1} - b^n$$

$$\Rightarrow \quad a^{n-1}(a - b) = b^{n-1}(a - b)$$

$$\Rightarrow \quad a^{n-1} = b^{n-1}$$

$$\Rightarrow \quad \frac{a^{n-1}}{b^{n-1}} = 1$$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^{n-1} = \left(\frac{a}{b}\right)^0$$

$$\Rightarrow \quad n - 1 = 0$$

$$\Rightarrow \quad n = 1$$

Example 1.5: Between 1 and 31, m arithmetic means are inserted so that the ratio of the 7th and $(m - 1)^{th}$ means is 5: 9. Find the value of m. [NCERT]

Ans. Let $A_1, A_2, \ldots A_m$ be m arithmetic means between 1 and 31. Then 1, A_1, $A_2 \ldots A_m$, 31 is an A.P. with common difference d given by

$$d = \frac{31-1}{m+1} = \frac{30}{m+1}$$

Now, $$A_7 = 1 + 7d = 1 + \frac{7\times30}{m+1} = \frac{m+211}{m+1}$$

And, $$A_{m-1} = 1 + (m - 1)d$$
$$= 1 + \frac{30(m-1)}{m+1} = \frac{31m-29}{m+1}$$

It is given that

$$\frac{A_7}{A_{m-1}} = \frac{5}{9}$$

$$\frac{m+211}{31m-29} = \frac{5}{9}$$

$$\Rightarrow \quad 9m + 1899 = 155m - 145$$

$$\Rightarrow \quad 146m = 2044$$

$$\Rightarrow \quad m = 14$$

Example 1.6: In an A.P if m^{th} term is n and the n^{th} term is m, where $m \neq n$, find the p^{th} term. [NCERT]

Ans. Let a be the first term and d be the common difference of the given A.P.

Given that the m^{th} term is n and the n^{th} term is m, where $m \neq n$.

$\therefore \quad a_m = n$

$\Rightarrow \quad a + (m-1)d = n$ ...(i)

$\therefore \quad a_n = m$

$\Rightarrow \quad a + (n-1)d = m$ ...(ii)

On subtracting (ii) from (i), we get

$(m-1)d - (n-1)d = n - m$

$\Rightarrow \quad (m - 1 - n + 1)d = n - m$

$\Rightarrow \quad d = -1$ ...(iii)

Putting value of d in (i), we get

$a - (m-1) = n$

$\Rightarrow \quad a = n + m - 1$ ...(iv)

So, the p^{th} term of the A.P. is given by

$a_p = a + (p-1)d$

$= (n + m - 1 + (p-1)(-1)$

$= n + m - 1 - p + 1 = n + m - p$

[using (iii) and (iv)]

TOPIC 3

GEOMETRIC PROGRESSION

A sequence $a_1, a_2, a_3, ..., a_n$ is called a geometric progression. If each term is non-zero and follows a relation $\frac{a_{k+1}}{a_k} = r$ where r is a constant for all $k \in N$

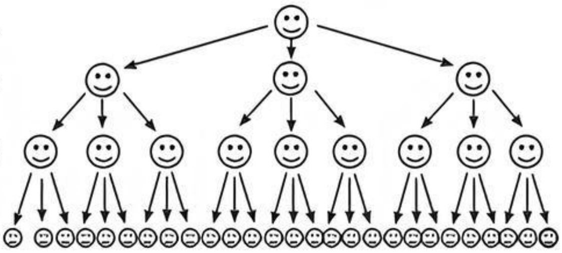
$\Rightarrow$ "r" is called the common ratio.

Geometric progressions happen whenever each agent of a system acts independently.

Here are some examples:

(i) The amount on your savings account;

(ii) The amount of money in your piggy bank if you deposit the same amount each week (a bank account with regular deposits leads you to arithmetic-geometric sequences).



General Term of a G.P.

Let us consider a G.P. with first non-zero term 'a' and common ration 'r'. Write a few terms of it. The second term is obtained by multiplying a by r, thus $a_2 = ar$. Similarly, third term is obained by multiplying a_2 by r. Thus, $a_3 = a_2 r = ar^2$, and so on.

We write below these and few more terms.

1st term $= a_1 = a = ar^{1-1}$,

2nd term $= a_2 = ar = ar^{2-1}$,

3rd term $= a_3 = ar^2 = ar^{3-1}$

4th term $= a_4 = ar^3 = ar^{4-1}$,

5th term $= a_5 = ar^4 = ar^{5-1}$

Do you see a pattern? What will be 16th term?

$a^{16} = ar^{16-1} = ar^{15}$

Thus, a G.P. can be written as $a, ar, ar^2, ar^3, ... ar^{n-1}$; $a, ar, ar^2, ..., ar^{n-1}$ ...., according as G.P. is finite or infinite, repsectively.

The seires $a + ar + ar^2 + ... + ar^{n-1}$ or $a + ar + ar^2 + ... + ar^{n-1} + ...$ are called finite or infinite geometric series, respectively.

Properties of G.P

In this section, we will discuss important properties of geometric progressions and geometric series.

Property I

If all the terms of G.P. are multiplied or divided by the same non-zero constant, then the resulting sequence is also a G.P. with the same common ratio.

Proof:

Let $a_1, a_2, a_3, ..., a_n$ be in geometric progression with common ratio 'r'.

$\frac{a_{n+1}}{a_n}$, for all $n \in N$ ...(i)

Let 'k' be any non-zero constant by which we are multiplying each term of G.P. So, we get

$ka_1, ka_2, ka_3, ..., ka_n$

So, equation (i) $\Rightarrow \frac{ka_{n+1}}{ka_n}$, for all $n \in N$

which is same as $\frac{a_{n+1}}{a_n} = r$.

New sequence formed is also in G.P. with common ratio 'r'.

Property II

The reciprocal of the terms of a given G.P. also form a G.P.

Proof:

Let $a_1, a_2, a_3, ..., a_n$ be in geometric progression with common ratio 'r'.

$\frac{a_{n+1}}{a_n}$, for all $n \in N$ ...(i)

Now the sequence formed by reciprocal will be

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}.$$

The common ratio

$$= \frac{\frac{1}{a_{n+1}}}{\frac{1}{a_n}} = \frac{a_n}{a_{n+1}} = \frac{1}{r}$$

So, the new sequence is in G.P. with common ratio $\frac{1}{r}$.

Property III

If each term of G.P. is raised by the same exponent then the resulting sequence is also a G.P.

Proof:

Let $a_1, a_2, a_3, \ldots, a_n$ be in geometric progression with common ratio 'r'.

$\frac{a_{n+1}}{a_n}$, for all $n \in N$ ...(i)

Let each term of G.P. is raised by exponent, 'K', then the sequence becomes $a_1^k, a_2^k, a_3^k, \ldots, a_n^k$

$$\frac{a_{n+1}^k}{a_n^k} = \left(\frac{a_{n+1}}{a_n}\right)^k = r^k \text{ for all } n \in N$$

$\therefore$ $a_1^k, a_2^k, a_3^k, \ldots, a_n^k$ is in G.P. with common ratio r^k.

Property IV

In a finite G.P., the product of the terms equidistant from the beginning and from the end is always same and is equal to the product of the first and the last term.

Proof:

Let $a_1, a_2, a_3, \ldots, a_n$ be in geometric progression with common ratio 'r'.

K^{th} term from the beginning $= a_k = a_1 r^{k-1}$

K^{th} term from the end $= a_{n-k+1} = a_1 r^{n-k}$

$\therefore$ Product of k^{th} term from the beginning and k^{th} term from the end

$$= a_1 r^{k-1} \times a_1 r^{n-k}$$

$$= a_1^2 r^{k-1+n-k}$$

$$= a_1^2 r^{n-1} \text{ for } K = 2, 3, 4, \ldots (n-1)$$

Hence, the product of the terms equidistant from the beginning and the end is always same and is equal to the product of first and the last term.

Property V

Three non-zero numbers a, b, c are in G.P. if and only if $b^2 = ac$.

Proof: a, b, c are in G.P.

$\therefore$ Common ratio $= \frac{b}{a} = \frac{c}{b}$

$\therefore$ $b^2 = ac$

Property VI

If the terms of a given G.P. are chosen at regular intervals, then the new sequence forms a G.P.

Property VII

If $a_1, a_2, a_3, \ldots, a_n$ be in geometric progression, then $\log a_1, \log a_2, \log a_3, \ldots, \log a_n$ is an arithmetic progression and vice-versa.

Sum of n terms of a G.P.

If a and r are the first term and common ratio of a G.P. respectively, then the sum of n terms of this G.P. is given by

$$a_n = ar^{n-1}$$

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) \text{ if } r > 1$$

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right) \text{ if } r < 1$$

Sum of infinite terms

$$S_n = a_n \text{ if } r = 1$$

Sum of GP for Infinite Terms

If the number of terms in a GP is not finite, then the GP is called infinite GP. The formula to find the sum to infinity of the given GP is:

$$S_\infty = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}; -1 < r < 1$$

Where, S_∞ = Sum of infinite geometric progression

a = First term of G.P.

r = Common ratio of G.P.

n = Number of terms

Geometric Mean

(1) If a, b and c are in G.P., then, b is called the geometric mean and it is given by

$$b = \sqrt{ab}$$

(2) If $G_1, G_2, G_3, \ldots, G_n$ are n numbers between positive numbers a and b then

$$\text{Common ratio, } r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

and $$G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Example 1.7: Show that the sequence 5, 10, 20, 40, ..., is a G.P. Write its general term. [NCERT]

Ans. Given, sequence is 5, 10, 20, 40, . . .

Since, $\frac{10}{5}=2, \frac{20}{10}=2, \frac{40}{20}=2, \ldots$

So, the given sequence is a G.P. with a common ratio, $r = 2$.

Also, the first term of G.P. is $a = 5$.

Hence, the general term of the G.P. is given by

$a_n = ar^{n-1} = 5(2^{n-1})$.

Example 1.8: If $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P. find x. [NCERT]

Ans. Given that $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P.

$$\therefore \quad \frac{x}{\left(-\frac{2}{7}\right)} = \frac{\left(-\frac{7}{2}\right)}{x}$$

$$\Rightarrow \quad x^2 = 1$$

$$\Rightarrow \quad x = \pm 1$$

Example 1.9: Show that the sequence whose n^{th} term is $a_n = 3^n$ is a G.P. Find its common ratio. [NCERT]

Ans. Given, $a_n = 3^n$

Then, $a^{n+1} = 3^{n+1}$

Now, $\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{3^n} = 3$, which is an independent of n.

Hence, the given sequence is a G.P. with a common ratio of 3.

Example 1.10: Find the number of terms in the G.P. 2, 6, 18, ...,13122. [NCERT]

Ans. Given G.P. is 2, 6, 18, ..., 13122

Since, $\frac{6}{2}=3, \frac{18}{6}=3, \ldots$

So, the common ratio of the given G.P. is $r = 3$.

Also, the first term of the G.P. is $a = 2$.

Let the number of terms in the given G.P. be n.

Then,

$$a_n = 13122$$

$$\Rightarrow \quad ar^{n-1} = 13122$$

$$\Rightarrow \quad 2(3^{n-1}) = 13122 \quad [\because a = 2, r = 3]$$

$$\Rightarrow \quad 3^{n-1} = 6561$$

$$\Rightarrow \quad 3^{n-1} = 3^8$$

$$\Rightarrow \quad n - 1 = 8$$

$$\Rightarrow \quad n = 9$$

Hence, the number of terms in the given G.P. is 9.

Example 1.11: If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P. [NCERT]

Ans. Let a be the first term and r be the common ratio of the given G.P.

Given that 4^{th}, 10^{th} and 16^{th} terms of the G.P. are x, y and z respectively.

$$\therefore \quad a_4 = x$$

$$\Rightarrow \quad ar^3 = x$$

$$\therefore \quad a_{10} = y$$

$$\Rightarrow \quad ar^9 = y$$

$$\therefore \quad a_{16} = z$$

$$\Rightarrow \quad ar^{15} = z$$

Since, $\frac{y}{x} = \frac{ar^9}{ar^3} = r^6$ and $\frac{z}{y} = \frac{ar^{15}}{ar^9} = r^6$

$$\therefore \quad \frac{y}{x} = \frac{z}{y}$$

Hence, x, y, z are in G.P.

Example 1.12: If, $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx} (x \neq 0)$ then show that a, b, c and d are in G.P. [NCERT]

Ans. We have,

$$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$$

Applying componendo- dividendo

$$\frac{a+bx+a-bx}{(a+bx)-(a-bx)} = \frac{b+cx+(b-cx)}{b+cx-(b-cx)} = \frac{c+dx+(c-dx)}{c+dx-(c-dx)}$$

$$\frac{a+a+bx-bx}{bx+bx-a+a} = \frac{b+b+cx-cx}{cx+cx-b+b} = \frac{c+dx+c-dx}{dx+dx-c+c}$$

$$\frac{2a+0}{2bx+0} = \frac{2b+0}{2cx+0} = \frac{2c+0}{2dx+0}$$

$$\frac{2a}{2bx} = \frac{2b}{2cx} = \frac{2c}{2dx}$$

$$\frac{a}{bx} = \frac{b}{cx} = \frac{c}{dx}$$

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$$

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

As the common ratio is same, thus a, b, c and d are in G.P.

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. The n^{th} term of a G.P 5, 25, 125, ...is:

(a) 5^n (b) 5^{n-1}

(c) 5^{n+1} (d) 5^{n-2}

Ans. *(a)* 5^n

Explanation: Here;

$$a = 5$$
$$r = 5$$

So, $a_n = ar^{n-1}$

$= 5(5^{n-1})$

$= 5^n$

2. If the third term of G.P. is 4, then the product of its first 5 terms is:

(a) 4^3 (b) 4^4

(c) 4^5 (d) none of these

[Diksha]

Ans. *(c)* 4^5

Explanation: Here,

$$a_3 = 4$$
$$ar^2 = 4$$

Product of first 5 terms

$= a.\ ar.\ ar^2.\ ar^3.\ ar^4$

$= a^5.\ r^{10}$

$= (ar^2)^5$

$= (4)^5$

3. In a G.P, the 3^{rd} is 24 and the 6^{th} term is 192, then the 10^{th} term is:

(a) 1084 (b) 3290

(c) 3072 (d) 2340

Ans. *(c) 3072*

Explanation: Here,

$a_3 = 24$ and $a_6 = 192$

So, $ar^2 = 24$ and $ar^5 = 192$

On dividing, we get

$$r^3 = 8$$
$$r = 2, a = 6$$
$$a_{10} = ar^9$$

$= (6.\ 2^9)$

$= 3072$

4. If the 8^{th} term of G.P is 192 with a common ratio of 2, then the 12^{th} term is:

(a) 1640 (b) 2084

(c) 3072 (d) 3126

Ans. *(c) 3072*

Explanation: Here,

$$T_8 = 192$$

and $r = 2$

We know that

$$T_n = ar^{n-1}$$

So, $T_8 = ar^{8-1}$

$$ar^7 = 192$$
$$a2^7 = 192$$
$$a = \frac{192}{128}$$
$$a = \frac{3}{2}$$

On putting $a = \frac{3}{2}$ and $n = 12$ in $T_n = ar^{n-1}$, we get

$$T_{12} = ar^{12-1}$$
$$T_{12} = ar^{11}$$

$= \frac{3}{2} \times 2^{11}$

$= 3 \times 2^{10}$

$= 3 \times 1024 = 3072$

5. Which term of the G.P 5, 10, 20, 40,.... is 5120?

(a) 11^{th} (b) 10^{th}

(c) 6^{th} (d) 5^{th}

Ans. *(a)* 11^{th}

Explanation: Given sequence 5, 10, 20, 40, ... is a G.P.

Here $a = 5$, $r = 2$ and $a_n = 5120$

As we know that the general term of a GP is given by

$$a_n = ar^{n-1}$$
$$5120 = 5.\ 2^{n-1}$$
$$2^{n-1} = \frac{5120}{5}$$
$$2^{n-1} = 1024$$
$$2^{n-1} = 2^{10}$$
$$n - 1 = 10$$
$$n = 11$$

Hence, 11^{th} term is 5120.

6. The 5^{th} term from the end of the sequence 16, 8, 4, 2 ... $\frac{1}{16}$ is:

(a) 1 (b) 2

(c) 3 (d) 4

Ans. *(a) 1*

Explanation: Given sequence is 16, 8, 4, 2, ... $\frac{1}{16}$.

Here,

$$a = 16 \text{ and}$$
$$r = \frac{1}{2}$$

We know that

$$a_n = ar^{n-1}$$

So, $n = 5^{th}$ term

$$a_5 = ar^4$$

On putting $r = \frac{1}{2}$ we get

$$= 16 \times \left(\frac{1}{2}\right)^4$$
$$= 1$$

7. **18^{th} term from the end of the sequence 3, 6, 12,...25^{th} term is:**

(a) 393216 (b) 393206
(c) 313216 (d) 303216

Ans. *(a) 393216*

Explanation: Here, $a = 3$, $r = 2$

$m = 18$ and $n = 25$

We know that if a sequence has n term then m^{th} term from end is equal to $(n - m + 1)$.

$$a_{18} = ar^{18-1}$$
$$= 3(2)^{18-1}$$
$$= 3\ (2)^{17}$$
$$= 3 \times 131072$$
$$= 393216$$

8. **If n terms of a G.P. 3, 3^2, 3^3... are needed to give the sum 120, then the value of n is:**

(a) 2 (b) 3
(c) 4 (d) 5

Ans. *(c) 4*

Explanation: Here, $S_n = 120$, $a = 3$ and $r = 3$

So,

$$\therefore \quad S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$
$$20 = 3\left(\frac{3^n - 1}{3 - 1}\right)$$
$$240 = 3(3^n - 1)$$
$$80 = 3^n - 1$$
$$81 = 3^n$$
$$3^4 = 3^n$$
$$n = 4$$

9. **If x, $2y$, $3z$ are in A.P., where the distinct numbers x, y, z are in G.P., then the common ratio of the G.P. is:**

(a) 3 : 1 (b) 1 : 3
(c) 2 : 1 (d) 1 : 2

[NCERT Exemplar]

Ans. *(b) 1 : 3*

Explanation: Since, x, $2y$, $3z$ are in A.P.

$$\therefore \quad 2y - x = 3z - 2y$$
$$\Rightarrow \quad 4y = x + 3z \quad \text{...(i)}$$

Now, x, y, z are in G.P.

$$\therefore \quad \text{common ratio } r = \frac{y}{x} = \frac{z}{y} \quad \text{...(ii)}$$
$$\therefore \quad y^2 = xz$$

On putting the value of x from eq. (i), we get

$$y^2 = (4y - 3z)z$$
$$\Rightarrow \quad y^2 = 4yz - 3z^2$$
$$\Rightarrow \quad 3z^2 - 4yz + y^2 = 0$$
$$\Rightarrow \quad 3z^2 - 3yz - yz + y^2 = 0$$
$$\Rightarrow \quad 3z\,(z - y) - y(z - y) = 0$$
$$\Rightarrow \quad (3z - y)\,(z - y) = 0$$
$$\Rightarrow \quad 3z - y = 0 \text{ and } z - y = 0$$
$$\Rightarrow \quad 3z = y \text{ and } z \neq y$$

[$\because$ z and y are distinct numbers]

$$\Rightarrow \quad \frac{z}{y} = \frac{1}{3} \quad \text{[from eq. (ii)]}$$
$$\Rightarrow \quad r = \frac{1}{3}$$

10. **The lengths of three unequal edges of a rectangular solid block are in G.P. If the volume of the block is 512 cm^3 and the total surface area is 232 cm^2, then the length of the longest edge is:**

(a) 8 cm (b) 12 cm
(c) 9 cm (d) 15 cm

Ans. *(a) 8 cm*

Explanation: Let the three unequal length of the block be $\frac{A}{r}$, A and Ar.

The volume A^3 equals to 512.

$$\Rightarrow \quad A^3 = 512$$
$$\Rightarrow \quad A = 8$$

Also, the total surface area $= 2\left(\frac{A^2}{r} + A^2 + A^2 r\right)$

Hence, $2\left(\frac{A^2}{r} + A^2 + A^2 r\right) = 232$

$$\Rightarrow \quad \frac{1}{r} + 1 + r = \frac{232}{2A^2}$$
$$= \frac{232}{2 \times 64} = \frac{29}{16}$$

$$\Rightarrow \quad \frac{1}{r}+1+r=\frac{29}{16}$$

$$\Rightarrow \quad \frac{1}{r}+r=\frac{29}{16}-1$$

$$\Rightarrow \quad \frac{1}{r}+r=\frac{13}{16}$$

Hence, $r = 1$ or $\frac{13}{16}$

So, the longest side = 8 × 1 = 8 *cm*

11. **If the third term of G.P. is 4, then the product of its first 5 terms is:**

(a) 4^3 (b) 4^4

(c) 4^5 (d) 4^6 [Diksha]

Ans. *(c)* 4^5

Explanation: The n^{th} term of a G.P. is ar^{n-1}.

The third term of a G.P. is 4.

$$ar^2 = 4$$

So, the product of the first 5 terms is,

$$a(ar)(ar^2)(ar^3)(ar^4) = a^5r^{10}$$
$$= (ar^2)^5$$
$$= 4^5$$

Assertion Reason Questions

Direction: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R).

Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

12. **Assertion (A): If 5^{th} and 8^{th} term of a *G.P* be 48 and 384 respectively, then the common ratio of *G.P* is 2.**

Reason (R): If 18, *x*, 14 are in A.P, then $x = 16$.

Ans. *(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).*

Explanation: We have,

$$T_5 = 48$$

We know that

$$\therefore \quad ar^4 = 48 \quad ...(i)$$
$$T_8 = 384$$
$$T_9 = ar^{n-1}$$
$$ar^7 = 384 \quad ...(ii)$$

So, on dividing eq. (ii) by (i), we get

$$\frac{ar^7}{ar^4} = \frac{384}{48}$$
$$r^3 = 8$$
$$r = 2$$

Given, 18, *n*, 14 are in A.P

So,

$$\text{Common ratio} = n - 18$$
$$= 14 - n$$
$$2n = 32$$
$$n = 16$$

13. **Assertion (A): The sum of the first 20 terms of an A.P, 4, 8, 12 is equal to 840.**

Reason (R): Sum of *n* terms of an A.P is $\left[S_n = \frac{n}{2}(2a+(n-1)d\right]$.

Ans. *(a) Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: Given A.P. is 4, 8, 12

$a = 4$ and $d = 4$

We know that

$$S_n = \frac{n}{2}\ 2a+(n-1)d$$

Now, $$S_{20} = \frac{20}{2}\{2\times4+(20-1)\times4\}$$
$$S_{20} = 10(8 + 76)$$
$$= 10 \times 84$$
$$= 840$$

14. **Assertion (A): The sum of the series $\frac{3}{\sqrt{5}}+\frac{4}{\sqrt{5}}+\sqrt{5}+ ...$ 25 term is $75\sqrt{5}$.**

Reason (R): If 27, *n*, 3 are in G.P then $x = \pm 4$.

Ans. *(c) (A) is true but (R) is false.*

Explanation:

Given, $$S_n = \frac{3}{\sqrt{5}}+\frac{4}{\sqrt{5}}+\sqrt{5}+\ldots$$

Here, $a = \frac{3}{\sqrt{5}}$ and $d = \frac{1}{\sqrt{5}}$

So, $$S_n = \frac{25}{2}\left[2\times\frac{3}{\sqrt{5}}+(25-1)\frac{1}{\sqrt{5}}\right]$$
$$= 25\times\frac{15}{5}\times\sqrt{5} = 75\sqrt{5}$$

Given, 27, n, 3 are in G.P.

$$\frac{n}{27}=\frac{3}{n}$$

So, $n^2 = 81$

$\Rightarrow$ $n = \pm 9$

15. Assertion (A): If the number $\frac{-2}{7}, k, \frac{-7}{2}$ is in G.P then $k = \pm 1$.

Reason (R): If a_1, a_2, a_3 are in G.P then $\frac{a_2}{a_1}=\frac{a_3}{a_2}$.

Ans. (a) *Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: We know that, if a_1, a_2, a_3 are in G.P. then $\frac{a_2}{a_1} = \frac{a_3}{a_2}$

If $\frac{-2}{7}, k, \frac{-7}{2}$ are in G.P.

Then, $\frac{a_2}{a_1}=\frac{a_3}{a_2}$

$$\frac{k}{-\frac{2}{7}}=\frac{-\frac{7}{2}}{k}$$

$14k^2 = 14$

$k^2 = 1$

$k = \pm 1$

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

16. *A.M. and G.M. Inequality*

Let a, b be two positive real numbers, then

$$(\sqrt{a}-\sqrt{b})^2 \geq 0$$

$\Rightarrow$ $a+b-2\sqrt{ab} \geq 0$

$\Rightarrow$ $\frac{a+b}{2} \geq \sqrt{ab}$

$\Rightarrow$ *A.M $\geq$ G.M*

Equality holds iff $a = b$

i.e., A.M. = G.M. iff $a = b$

The A.M. and G.M. inequality for two variables can be generalised to n variables.

If $a_1, a_2, a_3 ... a_n$ are n positive real numbers, then

$$\frac{a_1+a_2+a_3+...+a_n}{n} \geq (a_1.a_2.a_3 ... a_n)^{\frac{1}{n}}.$$

(A) If x, y, z are positive integers, then find value of the expression $(x + y)(y + z)(z + x)$.

(B) Find the minimum value of the expression $3^x + 3^{1-x}$, $x \in R$ and if three positive real numbers a, b, c are in A.P. and $abc = 4$, then find the minimum possible value of b.

(C) Find the common ratio of a GP whose sum of infinite terms is 8 and its second term is 2.

Ans. (A) $\because$ A.M. > G.M.

$\therefore$ $\frac{x+y}{2} > \sqrt{xy}$

$\frac{y+z}{2} > \sqrt{yz}$

And $\frac{z+x}{2} > \sqrt{zx}$

On multiplying three inequalities, we get

$$\frac{x+y}{2}.\frac{y+z}{2}.\frac{z+x}{2} > \sqrt{(xy)(yz)(xz)}$$

$\Rightarrow$ $(x + y)(y + z)(z + x) > 8xyz$

(B) $\because$ A.M. $\geq$ G.M.

$\therefore$ $\frac{a_1+a_2+a_3+...+2a_n}{n} \geq (a_1.a_2.a_3...2a_n)^{\frac{1}{n}}$

$\Rightarrow$ $\frac{3^x+3^{1-x}}{2} \geq \sqrt{3^x.3^{1-x}}$

$\Rightarrow$ $3^x+3^{1-x} \geq 2\sqrt{3}$

So, the minimum value of 3^x+3^{1-x} is $2\sqrt{3}$.

Given a, b, c are in A.P.

So, $2b = a + c$

and $abc = 4$

We know that, A.M. $\geq$ G.M.

So, $\frac{a+b+c}{3} \geq (abc)^{1/3}$

$\Rightarrow$ $\frac{2b+b}{3} \geq (4)^{1/3}$ [using (i) and (ii)]

$\Rightarrow$ $\frac{3b}{3} \geq (2^2)^{1/3}$

$\Rightarrow$ $b \geq 2^{\frac{2}{3}}$

So, the minimum possible value of b is $2^{\frac{2}{3}}$.

(C) Let a be the first term and r be the common ratio of the G.P.

Given, $ar = 2$ and $S_{\infty} = 8 = \frac{a}{1-r}$

$$\Rightarrow \quad 8 = \frac{2}{r(1-r)} \qquad \left[\because a = \frac{2}{r}\right]$$

$$\Rightarrow \quad 4r(1-r) = 1$$

$$\Rightarrow \quad 4r - 4r^2 - 1 = 0$$

$$\Rightarrow \quad 4r^2 - 4r + 1 = 0$$

$$\Rightarrow \left(r - \frac{1}{2}\right)(4r - 2) = 0$$

$$\Rightarrow \quad r = \frac{1}{2}$$

17. *A sequence of non-zero numbers is said to be a geometric progression, if the ratio of each term, except the first one, by its preceding term is always constant. Rahul, being a plant lover, decides to open a nursery and he bought few plants and pots. He wants to place pots in such a way that the number of pots in the first row is 2, in second row is 4 and in the third row is 8 and so on....*

(A) The constant multiple by which the number of pots is increasing in every row is:

(a) 2 (b) 4

(c) 8 (d) 1

(B) The number of pots in 8^{th} row is:

(a) 156 (b) 256

(c) 300 (d) 456

(C) The difference in number of pots placed in 7^{th} row and 5^{th} row is:

(a) 86 (b) 50

(c) 90 (d) 96

(D) Total number of pots upto 10^{th} row is:

(a) 1046 (b) 2046

(c) 1023 (d) 1024

(E) If Rahul wants to place 510 pots in total, then the total number of rows formed in this arrangement is:

(a) 7 (b) 8

(c) 9 (d) 5

Ans. (A) *(a) 2*

Explanation: The number of pots in each row is 2, 4, 8.... .

$\therefore$ This forms a geometric progression.

Where $a = 2, r = \frac{4}{2} = 2$

Hence, the constant multiple by which the number of pots is increasing in every row is 2.

(B) *(b) 256*

Explanation: Number of pots in 8^{th} row $= a_8$

$$a_8 = ar^{8-1} = 2(2)^7 = 2^8 = 256$$

(C) *(d) 96*

Explanation: Number of pots in 7^{th} row,

$$a_7 = 2(2)^{7-1} = 2.2^6 = 2^7 = 128$$

Number of pots in 5^{th} row,

$$a_5 = 2(2)^{5-1} = 2.2^4 = 2^5 = 32$$

Required answer = 128 – 32 = 96

(D) *(b) 2046*

Explanation: Total number of pots upto 10^{th} row

$$\therefore \quad S_{10} = \frac{a(r^{10}-1)}{r-1} = \frac{2(2^{10}-1)}{2-1}$$

$$= \frac{2(1024-1)}{1} = 2046$$

(E) *(b) 8*

Explanation: Let there be n number of rows.

$$\therefore \quad S_n = 510 = \frac{2(2^n-1)}{2-1}$$

$$\Rightarrow \quad \frac{510}{2} = 2^n - 1$$

$$\Rightarrow \quad 255 = 2^n - 1$$

$$\Rightarrow \quad 256 = 2^n$$

$$\Rightarrow \quad 2^8 = 2^n$$

$$\Rightarrow \quad n = 8$$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

18. Find the sum of the G.P. 0.15, 0.015, 0.0015, ... to 20 terms. **[NCERT Exemplar]**

Ans. Given G.P. is 0.15, 0.015, 0.0015, . . .

Since, $\frac{0.015}{0.15}=0.1, \frac{0.0015}{0.015}=0.1, ...$

So, the common ratio of the given G. P is $r = 0.1$.

Also, the first term of the G.P. is $a = 0.15$.

Hence, the sum of 20 terms of the G.P. is given by

$$\therefore \quad S_{20}=\frac{a(1-r^{20})}{1-r}$$

$$=\frac{(0.15)[1-(0.1)^{20}]}{1-0.1}$$

$$=\frac{1}{6}[1-(0.1)^{20}]$$

19. If a, b and c are in G.P. then find the value of $\frac{a-b}{b-c}$. **[Delhi Gov. Term-1 SQP 2021]**

Ans. Since, a, b and c are in G.P.

$$\therefore \quad \frac{b}{a}=\frac{c}{b}=r \quad \text{(constant)}$$

$\Rightarrow$ $b = ar$ and $c = br$

$\Rightarrow$ $c = ar.\ r = ar^2$

So, $\frac{a-b}{b-c}=\frac{a-ar}{ar-ar^2}=\frac{a(1-r)}{ar(1-r)}$

$$\frac{a-b}{b-c}=\frac{1}{r}$$

Hence, the correct value of $\frac{a-b}{b-c}$ is $\frac{a}{b}$ or $\frac{b}{c}$.

20. Find the sum of the G.P. 1, – a, a^2, – a^3, ... to n terms.

Ans. Given G.P. is 1, – a, a^2, – a^3,...

Since, $\frac{-a}{1}=-a, \frac{a^2}{-a}=-a, ...$

So, the common ratio of the given G.P. is $r = -a$.

Also, the first term of the G.P. is $A = 1$.

We know that, the sum of n terms of the G.P. is given by

$$\therefore \quad S_n=\frac{A(1-r^n)}{1-r}$$

$$=\frac{(1)[1-(-a)^n]}{1-(-a)}=\frac{1-(-a)^n}{1+a}$$

21. Which term of the sequence 3, 10, 17, ... is 136? **[Delhi Gov. QB 2022]**

Ans. Given sequence 3, 10, 17, ...

Let n^{th} term be the 136 i.e.,

$$T_n = a + (n - 1)d$$

$\Rightarrow$ $136 = 3(n - 1)7$ $\quad [\because a = 3 \text{ and } d = 7]$

$\Rightarrow$ $7n = 140$

$\Rightarrow$ $n = 20$

Hence, 20^{th} term is 136.

22. Find the sum to n terms of the G.P., whose k^{th} term is 5^k.

Ans. Given that k^{th} term of a G.P. is 5^k.

$\therefore \quad a_k = 5^k$ ...(i)

Putting $k = 1$ in (i), we get

$a_1 = 5$ ...(ii)

Putting $k = 2$ in (i), we get

$a_2 = 25$

Then, the first term is $a = a_1 = 5$ and the common ratio is $r=\frac{a_2}{a_1}=\frac{25}{5}=5$

We know that, the sum of n terms of the G.P. is given by

$$\therefore \quad S_n=\frac{a(r^n-1)}{r-1}$$

$$=\frac{(5)(5^n-1)}{5-1}=\frac{5}{4}(5^n-1)$$

23. Find the 9^{th} term and the general term of the progression: $\frac{1}{4}, -\frac{1}{2}, 1, -2, ...$

Ans. The given progression is clearly a G.P. with the first term $a = \frac{1}{4}$ and common ratio $r = -2$.

$\therefore$ 9^{th} term $= a_9 = ar^{9-1}$

$$= ar^8 = \frac{1}{4}(-2)^8 = 64$$

And,

General term $= a_n = ar^{n-1}$

$$= \frac{1}{4}(-2)^{n-1} = (-1)^{n-1}2^{n-3}$$

24. If in a G.P. $a_3 + a_5 = 90$ and if $r = 2$ find the first term of the G.P. **[Delhi Gov. QB 2022]**

Ans. Let a be the first term of the GP

Given,

$$r = 2 \text{ and } a_3 + a_5 = 90$$

$\Rightarrow \quad ar^2 + ar^4 = 90$

$\Rightarrow \quad a(2)^2 + a(2)^4 = 90$

$\Rightarrow \quad 4a + 16a = 90$

$\Rightarrow \quad 20a = 90$

$a = \frac{9}{2}$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

25. If the p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c respectively. Prove that $a^{q-r} b^{r-p} c^{p-q} = 1$.

Ans. Let A be the first term and R be the common ratio of the given G.P.

Given that p^{th}, q^{th} and r^{th} terms of the G.P are a, b and c respectively.

$\therefore \quad A_p = a$

$\Rightarrow \quad AR^{p-1} = a$ ...(i)

$\therefore \quad A_q = b$

$\Rightarrow \quad AR^{q-1} = b$ ...(ii)

$\therefore \quad A_r = c$

$\Rightarrow \quad AR^{r-1} = c$ ...(iii)

Hence, $a^{q-r} b^{r-p} c^{p-q} = (AR^{p-1})^{q-r} (AR^{q-1})^{r-p} (AR^{r-1})^{p-q}$

{Using (i), (ii) and (iii)}

$= A^{q-r+r-p+p-q} R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$

$= A^0 R^{(pq-q-pr+r+qr-r-pq+p+pr-p-qr+q)}$

$= A^0 R^0$

$= 1$

26. The fourth, seventh, and the last term of a G.P. are 10, 80 and 2560 respectively. Find the first term and the number of terms in the G.P.

Ans. Let a be the first term and r be the common ratio of the given G.P.

We know that, $a_n = ar^{n-1}$

$a_4 = 10$

$ar^3 = 10$ ...(i)

$a_7 = 80$

$ar^6 = 80$ ...(ii)

On dividing eq. (ii) by (i), we get

$a_n = ar^{n-1}$

$\Rightarrow \quad \frac{ar^6}{ar^3} = \frac{80}{10}$

$\Rightarrow \quad r^3 = 8$

$\Rightarrow \quad r = 2.$

Putting $r = 2$ in $ar^3 = 10$, we get

$\Rightarrow \quad a = \frac{10}{8} = \frac{5}{4}$

Let there be n terms in the given G.P.

Then, $a_n = 2560$

$\Rightarrow \quad ar^{n-1} = 2560$

$\Rightarrow \quad \frac{5}{4}(2^{n-1}) = 2560$

$\Rightarrow \quad 2^{n-4} = 256$

$\Rightarrow \quad 2^{n-4} = 2^8$

$\Rightarrow \quad n - 4 = 8$

$\Rightarrow \quad n = 12$

27. Which term of the G.P. 2, 1, $\frac{1}{2}$, $\frac{1}{4}$ .... is $\frac{1}{1024}$? [Delhi Gov. QB 2022]

Ans. Given, G.P. is $2, 1, \frac{1}{2}, \frac{1}{4}, ...$

First term $a = 2$ and common ratio $= \frac{1}{2}$

Let n^{th} term of G.P. be $\frac{1}{1024}$

$\therefore \quad T_n = ar^{n-1}$

$\Rightarrow \quad \frac{1}{1024} = 2\left(\frac{1}{2}\right)^{n-1}$

$\Rightarrow \quad \frac{1}{2048} = \left(\frac{1}{2}\right)^{n-1}$

$\Rightarrow \quad \left(\frac{1}{2}\right)^{11} = \left(\frac{1}{2}\right)^{n-1}$

$\therefore \quad n - 1 = 11$

or $\quad n = 12$

28. If the 4^{th} and 9^{th} term of a G.P. be 54 and 13122 respectively then find the G.P.

Ans. Let a be the first term and r the common ratio of the given G.P. Then,

$a_4 = 54$

$ar^3 = 54$ ...(i)

$a_9 = 13122$

$ar^8 = 13122$ ...(ii)

On dividing eq. (ii) by (i), we get

$\Rightarrow \quad \frac{ar^8}{ar^3} = \frac{13122}{54}$

$\Rightarrow \quad r^5 = 243$

$\Rightarrow \qquad r^5 = 3^5$

$\Rightarrow \qquad r = 3$

Putting $r = 3$ in $ar^3 = 54$, we get

$a(3)^3 = 54$

$\Rightarrow \qquad a = 2$

Thus, the given G.P. is $a, ar, ar^2, ar^3, ...$ i.e. 2, 6, 18, 54.

29. The 5^{th}, 8^{th} and 11^{th} terms of a G.P. are p, q and s respectively. Show that $q^2 = ps$.

Ans. Let a be the first term and r be the common ratio of the given G.P.

Given that 5^{th}, 8^{th} and 11^{th} terms of the G.P. are p, q, and s respectively.

$\therefore \qquad a_5 = p$

$\Rightarrow \qquad ar^4 = p \qquad$ ...(i)

$\therefore \qquad a_8 = q$

$\Rightarrow \qquad ar^7 = q \qquad$ ...(ii)

$\therefore \qquad a_{11} = s$

$\Rightarrow \qquad ar^{10} = s \qquad$ ...(iii)

Hence, $ps = (ar^4)(ar^{10})$ [Using (i) and (iii)]

$= a^2 r^{14}$

$= (ar^7)^2$

$= q^2$ [Using (ii)]

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

30. The n^{th} term of a G.P. is 128 and the sum of its n terms is 225. If its common ratio is 2, find the first term. **[Delhi Gov. QB 2022]**

Ans. From the question it is given that

The n^{th} term of a G.P. $T_n = 128$

The sum of its n terms $S_n = 255$

Common ratio $r = 2$

We know that, $T_n = ar^{n-1}$

$128 = a2^{n-1}$

$$a = \frac{128}{2^{n-1}} \qquad \text{...(i)}$$

Also we know that,

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

$$255 = a\frac{(2^n - 1)}{(2-1)}$$

By cross multiplication we get,

$255 = a(2^n - 1)$

$$a = \frac{255}{(2^n - 1)} \qquad \text{...(ii)}$$

Now, consider both the equation (i) and equation (ii)

$$\frac{255}{(2^n - 1)} = \frac{128}{(2^{n-1})}$$

By cross multiplication we get,

$255 \times 2^{n-1} = 128(2^n - 1)$

$255 \times 2^{n-1} = 128 \times 2^n - 128$

$$\frac{(255 \times 2^n)}{2} = 128 \times 2^n - 128$$

$255 \times 2^n = 256 \times 2^n - 256$

$256 \times 2^n - 255 \times 2^n = 256$

By simplification, $2^n = 256$

$2^n = 2^8$

By comparing both LHS and RHS, we get

Then, $128 = a2^7$

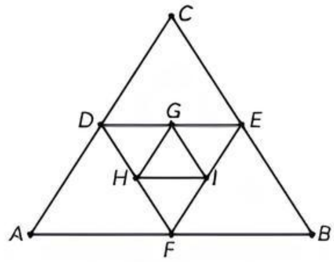
$128 = a \times 128$

$$a = \frac{128}{128}$$

$a = 1$

Therefore, the value of a is 1.

31. A side of an equilateral triangle is 20 cm long. A second equilateral triangle is inscribed in it by joining the mid-points of the sides of the first triangle. The process is continued as shown in the accompanying diagram. Find the perimeter of the sixth inscribed equilateral triangle. **[NCERT Exemplar]**



Ans. The side of the first equilateral ΔABC = 20 cm

By joining the mid-points of the sides of this triangle, we get the second equilateral triangle

of each side $= \frac{20}{2} = 10$ cm.

[$\because$ The line joining the mid-points of two sides of a triangle is $\frac{1}{2}$ and parallel to the third side of the triangle].

Similarly, each side of the third equilateral triangle $= \frac{10}{2} = 5$ cm

$\therefore$ Perimeter of first triangle $= 20 \times 3 = 60$ cm

Perimeter of the second triangle

$$= 10 \times 3 = 30 \text{ cm}$$

And the perimeter of the third triangle

$$= 5 \times 3 = 15 \text{ cm}$$

Therefore, the series will be 60, 30, 15, . . .

which is G.P. in which $a = 60$, and $r = \frac{30}{60} = \frac{1}{2}$

Now, we have to find the perimeter of the sixth inscribed equilateral triangle

$$\therefore \quad a_6 = ar^{6-1}$$

$$= 60 \times \left(\frac{1}{2}\right)^5$$

$$= 60 \times \frac{1}{32}$$

$$= \frac{15}{8} \text{ cm}$$

Hence, the required perimeter is $\frac{15}{8}$ cm.

32. If p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P., then show that $(p - q)$, $(q - r)$, $(r - s)$ are also in G.P.

Ans. Let a be the first term and d be the common difference of the given A.P. Further, let a_p, a_q, a_r and a_s be its p^{th}, q^{th}, r^{th} and s^{th} terms respectively. Then,

$$a_p = a + (p - 1)d, \quad a_q = a + (q - 1)d,$$

$$a_r = a + (r - 1)d \text{ and } a_s = a + (s - 1)d.$$

$$\Rightarrow \quad a_p - a_q = (p - q)d, \quad a_q - a_r = (q - r)d$$

and $a_r - a_s = (r - s)d$

It is given that a_p, a_q, a_r and a_s are in G.P. Let A be the first term and R be the common ratio of the G.P. Then,

$$A = a_p, AR = a_q, AR^2 = a_r \text{ and } AR^3 = a_s$$

$$\therefore \quad A - AR = a_p - a_q, AR - AR^2 = a_q - a_r$$

and $\quad AR^2 - AR^3 = a_r - a_s$

$$\Rightarrow \quad A(1 - R) = a_p - a_q, AR(1 - R) = a_q - a_r$$

and $\quad AR^2(1 - R) = a_r - a_s$

$$\Rightarrow \quad (a_q - a_r)^2 = \{AR(1 - R)\}^2$$

$$= \{A(1 - R)\}\{AR^2(1 - R)\}$$

$$= (a_p - a_q)(a_r - a_s)$$

$$\Rightarrow \quad (q - r)^2 d^2 = \{(p - q)d\}\{(r - s)d\}$$

$$\Rightarrow \quad (q - r)^2 = (p - q)(r - s)$$

$\Rightarrow$ $p - q$, $q - r$, $r - s$ are in G.P.

33. Which term of the G.P. $\sqrt{3}, 3, 3\sqrt{3}, ...$ is 729?

Ans. Given, G.P. is $\sqrt{3}, 3, 3\sqrt{3}, \ldots$

Since, $\frac{3}{\sqrt{3}} = \sqrt{3}, \frac{3\sqrt{3}}{3} = \sqrt{3}, \ldots$

So, the common ratio of the given G.P. is $r = \sqrt{3}$.

Also, the first term of the G.P. is $a = \sqrt{3}$

Let 729 be the n^{th} term of the G.P.

Then, $\quad a_n = 729$

$$\Rightarrow \quad ar^{n-1} = 729$$

$$\Rightarrow \quad \sqrt{3}(\sqrt{3})^{n-1} = 729 \quad [\because a = \sqrt{3}, r = \sqrt{3}]$$

$$\Rightarrow \quad (\sqrt{3})^n = 729$$

$$\Rightarrow \quad 3^{n/2} = 3^6$$

$$\Rightarrow \quad \frac{n}{2} = 6$$

$$\Rightarrow \quad n = 12$$

Hence, 729 is the 12^{th} term of the given G.P.

34. If the continued product of three numbers in G.P. is 216 and the sum of their products in pairs is 156. Find the numbers.

Ans. Let the three numbers be $\frac{a}{r}$, a, ar. Then,

$$\text{Product} = 216$$

$$\Rightarrow \quad \frac{a}{r}.(a).(ar) = 216$$

$$\Rightarrow \quad a^3 = 6^3$$

$$\Rightarrow \quad a = 6$$

Sum of the products in pairs = 156

$$\Rightarrow \quad \frac{a}{r}.a + a.ar + \frac{a}{r}.ar = 156$$

$$\Rightarrow \quad a^2\left(\frac{1}{r} + r + 1\right) = 156$$

$$\Rightarrow \quad 36\left(\frac{1 + r^2 + r}{r}\right) = 156$$

$$\Rightarrow \quad 3(r^2 + r + 1) = 13r$$

$$\Rightarrow \quad 3r^2 - 10r + 3 = 0$$

$$\Rightarrow \quad (3r - 1)(r - 3) = 0$$

$$\Rightarrow \quad r = \frac{1}{3} \text{ or, } r = 3$$

Putting the values of a and r, the required numbers are 18,6,2 or 2,6,18.

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

35. The length of three unequal edges of a rectangular solid block are in G.P. The volume of the block is 216 cm^3 and the total surface area is 252cm^2. Find the length of the largest edge. [NCERT Exemplar]

Ans. Let the length, breadth and height of a rectangular block be $\frac{a}{r}$, a and ar.

[Since they are in G.P.]

$\therefore$ Volume $= l \times b \times h$

$$216 = \frac{a}{r} \times a \times ar$$

$\Rightarrow$ $a^3 = 216$

$\Rightarrow$ $a = 6$

Now total surface area = $2[lb + bh + lh]$

$$252 = 2\left[\frac{a}{r}\cdot a + a.ar + \frac{a}{r}\cdot ar\right]$$

$$\Rightarrow \quad 252 = 2\left[\frac{a^2}{r} + a^2r + a^2\right]$$

$$\Rightarrow \quad 252 = 2a^2\left[\frac{1}{r} + r + 1\right]$$

$$\Rightarrow \quad 252 = 2 \times (6)^2\left[\frac{1+r^2+r}{r}\right]$$

$$\Rightarrow \quad 252 = 72\left[\frac{1+r^2+r}{r}\right]$$

$$\Rightarrow \quad \frac{252}{72} = \frac{1+r+r^2}{r}$$

$$\Rightarrow \quad \frac{7}{2} = \frac{1+r+r^2}{r}$$

$$2 + 2r + 2r^2 = 7r$$

$\Rightarrow$ $2r^2 - 5r + 2 = 0$

$\Rightarrow$ $2r^2 - 4r - r + 2 = 0$

$\Rightarrow$ $2r(r-2) - 1(r-2) = 0$

$\Rightarrow$ $(r-2)(2r-1) = 0$

$\Rightarrow$ $r - 2 = 0$ and $2r - 1 = 0$

$\therefore$ $r = 2, \frac{1}{2}$

Therefore, the three edges are:

If $r = 2$ then edges are 3, 6, 12.

If $r = \frac{1}{2}$ then edges are 12, 6, 3.

So, the length of the longest edge = 12

36. If the p^{th} and q^{th} terms of a G.P. are q and p respectively, then show that its $(p+q)^{th}$ term is $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$. [NCERT Exemplar]

Ans. Let a be the first term and r be the common ratio of a G.P.

Given that $a_p = q$

$\Rightarrow$ $ar^{p-1} = q$ ...(i)

and $a_q = p$

$\Rightarrow$ $ar^{q-1} = p$ ...(ii)

Dividing eq.(i) by eq. (ii), we get

$$\frac{ar^{p-1}}{ar^{q-1}} = \frac{q}{p}$$

$$\Rightarrow \quad \frac{r^{p-1}}{r^{q-1}} = \frac{q}{p}$$

$$\Rightarrow \quad r^{p-q} = \frac{q}{p}$$

$$\Rightarrow \quad r = \left(\frac{q}{p}\right)^{\frac{1}{p-q}}$$

Putting the value of r in eq. (i), we get

$$a\left[\frac{q}{p}\right]^{\frac{1}{p-q} \times p-1} = q$$

$$a\left[\frac{q}{p}\right]^{\frac{p-1}{p-q}} = q$$

$$\therefore \quad a = q.\left[\frac{p}{q}\right]^{\frac{p-1}{p-q}}$$

Now, $T_{p+q} = ar^{p+q-1}$

$$= q\left[\frac{p}{q}\right]^{\frac{p-1}{p-q}}\left[\frac{q}{p}\right]^{\frac{1}{p-q}(p+q-1)}$$

$$= \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \cdot \left(\frac{q}{p}\right)^{\frac{p+q-1}{p-q}}$$

$$= q\left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \cdot \left(\frac{p}{q}\right)^{\frac{-(p+q-1)}{p-q}}$$

$$= q\left(\frac{p}{q}\right)^{\frac{p-1}{p-q} - \frac{p+q-1}{p-q}}$$

$$= q\left(\frac{p}{q}\right)^{\frac{p-1-p-q+1}{p-q}}$$

$$= q\left(\frac{p}{q}\right)^{\frac{-q}{p-q}} = q\left(\frac{q}{p}\right)^{\frac{q}{p-q}}$$

$$= \frac{q^{\frac{q}{p-q}+1}}{p^{\frac{q}{p-q}}}$$

$$= \frac{q^{\frac{p}{p-q}}}{p^{\frac{q}{p-q}}}$$

$$= \left[\frac{q^p}{p^q}\right]^{\frac{1}{p-q}}$$

Hence, the required term is $\left[\frac{q^p}{p^q}\right]^{\frac{1}{p-q}}$.

37. Find the sum of 5 + 5.5 + 5.5.5 . . . to n terms.

Ans. Let S be the sum of the series 5 +5.5 + 5.55 + 5.555 + . . . to n terms.

S = 5 + 5.5 + 5.55 + 5.555 + . . . to n terms.

$\Rightarrow S = 5 + (5 + 0.5) + (5 + 0.55) + (5 + 0.555)$ + ... to n terms

$\Rightarrow S = (5 + 5 + 5 + \ldots$ to n terms$) + \{0.5 + 0.55 + 0.555 + \ldots$ to $(n-1)$terms$\}$

$\Rightarrow S = 5n + 5\{0.1 + 0.11 + 0.111 + \ldots$ to $(n-1)$ terms$\}$

$\Rightarrow S = 5n + \frac{5}{9}$ $\{0.9 + 0.99 + 0.999 +$ to $(n-1)$terms$\}$

$$\Rightarrow S = 5n + \frac{5}{9}\left\{\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right)+\ldots+\left(1-\frac{1}{10^{n-1}}\right)\right\}$$

$$\Rightarrow S = 5n + \frac{5}{9}\left\{(n-1)-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots+\frac{1}{10^{n-1}}\right)\right\}$$

$$\Rightarrow S = 5n + \frac{5}{9}\left[(n-1)-\frac{1}{10}\left\{\frac{1-\left(\frac{1}{10}\right)^{n-1}}{\left(1-\frac{1}{10}\right)}\right\}\right]$$

$$\Rightarrow S = 5n + \frac{5}{9}\left\{(n-1)-\frac{1}{9}\left(1-\frac{1}{10^{n-1}}\right)\right\}$$

38. If S be the sum, P the product, and R the sum of the reciprocals of n terms of a G.P., prove that $\left(\frac{S}{R}\right)^n = P^2$.

Ans. Let a be the first term and r be the common ratio of the G.P. Then,

$$S = a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{r^n-1}{r-1}\right)$$

$$P = a.\,ar\,ar^2 \ldots ar^{n-1}$$

$$= a^n r^{1+2+3+\ldots+(n-1)} = a^n r^{\frac{n(n-1)}{2}}$$

and,

$$R = \frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}+\ldots+\frac{1}{ar^{n-1}}$$

$$\Rightarrow R = \frac{1}{a}\frac{\left\{\left(\frac{1}{r}\right)^n-1\right\}}{\left\{\left(\frac{1}{r}\right)-1\right\}} = \frac{1}{a}\left(\frac{1-r^n}{1-r}\right)\frac{1}{r^{n-1}}$$

$$\Rightarrow R = \frac{1}{a}\left(\frac{r^n-1}{r-1}\right)\frac{1}{r^{n-1}}$$

$$\therefore \quad \frac{S}{R} = a\left(\frac{r^n-1}{r-1}\right).a\left(\frac{r-1}{r^n-1}\right)r^{n-1} = a^2r^{n-1}$$

$$\Rightarrow \left(\frac{S}{R}\right)^n = a^{2n}r^{n(n-1)} = \left\{a^n r^{\frac{n(n-1)}{2}}\right\}^{-2} = P^2$$

Hence, $\left(\frac{S}{R}\right)^n = P^2$

39. The product of first three terms of a G.P. is 1000. If 6 is added to its second term and 7 is added to its third term, the terms become in A.P. Find the G.P. [Delhi Gov. QB 2022]

Ans. Let the first three terms of the given GP be $\frac{a}{r}$, a and ar

According to the question, we get

Product of the first three terms of GP = 1000

$$\Rightarrow \quad \frac{a}{r}\times a \times ar = 1000$$

$$\Rightarrow \quad a^3 = (10)^3$$

$$\therefore \quad a = 10$$

It is given that $\frac{a}{r}$, $a + 6$, $ar + 7$ are in A.P.

$$\therefore \quad 2(a+6) = \frac{a}{r}+ar+7$$

[$\because$ If x, y, z are in AP, then $2y = x + z$]

$\Rightarrow \quad 2(10+6) = \frac{10}{r} + 10r + 7 \; [\because a = 10]$

$\Rightarrow \quad 32 = \frac{10}{r} + 10r + 7$

$\Rightarrow \quad 25 = \frac{10}{r} + 10r$

$\Rightarrow \quad 5 = \frac{2+2r^2}{r}$

$\Rightarrow \quad 2r^2 - 5r + 2 = 0$

$\Rightarrow \quad 2r^2 - r - 4r + 2 = 0$

$\Rightarrow \quad r(2r-1) - 2(2r-1) = 0$

$\Rightarrow \quad (2r-1)(r-2) = 0$

$\Rightarrow \quad r = 2, \frac{1}{2}$

Now, we have

$a = 10$ and $r = 2$ or $r = 2, \frac{1}{2}$

Taking $a = 10$ and $r = 2$, we get

$\left\{\frac{10}{2}, 10, 10 \times 2\right\}$

i.e., {5, 10, 20}

And also taking $a = 10$ and $r = \frac{1}{2}$, we get

$\left\{\frac{10}{\frac{1}{2}}, 10, 10 \times \frac{1}{2}\right\}$

i.e., {20, 10, 5}

Hence, the first three terms of GP are 5, 10, 20 for $r = 2$ and 20, 10, 5 for $r = \frac{1}{2}$.

RELATION BETWEEN A.M. AND G.M. 2

TOPIC 1

A.M. AND G.M.

Relation between Arithmetic Mean and Geometric Mean

The following results give the relation between arithmetic mean and geometric mean of two positive real numbers.

Result 1: The arithmetic mean of two positive real numbers is always greater than or equal to their geometric mean.

Proof: Let A and G denote the arithmetic mean and geometric mean of positive real numbers a and b.

Then, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Now, $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$

$$= \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{a}\sqrt{b}}{2}$$

$$= \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$$

Hence, $A \geq G$.

Result 2: The arithmetic mean of two positive real number is equal to their geometric mean if and only if they are equal.

Proof: Let A and G denote the arithmetic mean and geometric mean of positive real numbers a and b.

Then, by previous result, we have

$$A = G$$

$$\Leftrightarrow \quad A - G = 0$$

$$\Leftrightarrow \quad \frac{(\sqrt{a}-\sqrt{b})^2}{2} = 0$$

$$\Leftrightarrow \quad a = b$$

Example 2.1: If A.M. and G.M. of roots of a quadratic equation are 8 and 5 respectively, then obtain the quadratic equation. [NCERT]

Ans. Let a and b be the roots of a quadratic equation.

Given that A.M. and G.M. of roots of quadratic equation are A=8 and G=5 respectively.

We have,

$$A = 8$$

$$\Rightarrow \quad \frac{a+b}{2} = 8$$

$$\Rightarrow \quad a + b = 16 \qquad ...(i)$$

We have, $G = 5$

$$\Rightarrow \quad \sqrt{ab} = 5$$

$$\Rightarrow \quad ab = 25 \qquad ...(ii)$$

(On squaring both sides, we get)

Hence, the required quadratic equation is given by

$$x^2 - (\text{sum of roots})\, x + (\text{product of roots}) = 0$$

$$\Rightarrow x^2 - (a+b)\,x + ab = 0$$

$$\Rightarrow \quad x^2 - 16x + 25 = 0 \qquad \text{(Using (i) and (ii))}$$

Example 2.2: Show that the products of the corresponding terms of the sequences $a, ar, ar^2, ar^3 ... ar^{n-1}$ and $A, AR, AR^2, AR^3 ... AR^{n-1}$ from a G.P, and find the common ratio. [NCERT]

Ans. 1^{st} sequence is $a, ar, ar^2, ar^3 ... ar^{n-1}$

2^{nd} sequence is $A, AR, AR^2, AR^3 ... AR^{n-1}$

	1^{st} sequence	2^{nd} sequence	Product of corresponding terms
1^{st} term	a	A	$a \times A$
2^{nd} term	ar	AR	$ar \times AR$
3^{rd} term	ar^2	AR^2	$ar^2 \times AR^2$
n^{th} term	ar^{n-1}	AR^{n-1}	$ar^{n-1} \times AR^{n-1}$

Let P = Product of corresponding terms of 1^{st} and 2^{nd} sequence

$$= a \times A, ar \times AR, ar^2 \times AR^2, ..., ar^{n-1} \times AR^{n-1}$$

$$= aA, arAR, ar^2 AR^2, ..., aA(r^{n-1} R^{n-1})$$

$$P = aA, arAR, ar^2 AR^2, ..., aA(r^{n-1} R^{n-1})$$

Now, in $aA, arAR, ar^2 AR^2, ..., aA(r^{n-1} R^{n-1})$

$$\frac{arAR}{aA} = rR$$

and $$\frac{ar^2AR^2}{arAR} = Rr$$

Thus, $$\frac{\text{Second term}}{\text{First term}} = \frac{\text{Third term}}{\text{Second term}}.$$

i.e., common ratio is same

Thus, it is a G.P

$$\text{Common ratio} = r = \frac{\text{second term}}{\text{first term}}$$

$$= \frac{arAR}{aA}$$

$$= rR$$

Example 2.3: If S_1, S_2, S_3 are the sum of the first n natural numbers, their squared and their cubes respectively, show that $9S_2^2 = S_3(1 + 8S_1)$. [NCERT]

Ans. Given that S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes respectively.

$$S_1 = \frac{n(n+1)}{2}$$

$$S_2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3 = \left[\frac{n(n+1)}{2}\right]^2$$

Now, $\text{LHS} = 9S_2^2 = 9\left[\frac{n(n+1)(2n+1)}{6}\right]^2$

Or, $$9S_2^2 = \frac{n^2(n+1)^2(2n+1)^2}{4} \quad \text{...(i)}$$

Also, $\text{RHS} = S_3(1 + 8S_1)$

$$= \left[\frac{n(n+1)}{2}\right]^2 [1+4n(n+1)]$$

$$= \frac{n^2(n+1)^2}{4}[1+4n^2+4n]$$

Or, $$S_3(1 + 8S_1) = \frac{n^2(n+1)^2(2n+1)^2}{4} \quad \text{...(ii)}$$

From (i) and (ii), we get

$$9S_2^2 = S_3(1+8S_1)$$

Hence, proved.

Example 2.4: If A and G be $A.M$ and $G.M.$, respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$. [NCERT]

Ans. If a and b be two numbers

We need to show that the numbers are

$$A \pm \sqrt{(A+G)(A-G)}$$

i.e., $a = A+\sqrt{(A+G)(A-G)}$

$b = A-\sqrt{(A+G)(A-G)}$

Now we know that

Arithmetic mean $= A = \frac{a+b}{2}$

Geometric mean $= G = \sqrt{ab}$

Putting value of A and G in RHS we can prove that it is equal to a and b.

Solving

$$= A \pm \sqrt{(A+G)(A-G)}$$

$$= A \pm \sqrt{A^2 - G^2}$$

Putting $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

$$= \left(\frac{a+b}{2}\right) \pm \sqrt{\left(\frac{a+b}{2}\right)^2 - \left(\sqrt{ab}\right)^2}$$

$$= \left(\frac{a+b}{2}\right) \pm \sqrt{\frac{(a+b)^2}{4} - ab}$$

$$= \left(\frac{a+b}{2}\right) \pm \sqrt{\frac{a^2+b^2+2ab-4ab}{4}}$$

$$= \left(\frac{a+b}{2}\right) \pm \sqrt{\frac{a^2+b^2-2ab}{4}}$$

[$\because$ Using $(x-y)^2 = x^2 + y^2 - 2xy$]

$$= \frac{a+b}{2} \pm \sqrt{\frac{(a-b)^2}{4}}$$

$$= \frac{a+b}{2} \pm \sqrt{\left(\frac{a-b}{2}\right)^2}$$

$$= \frac{a+b}{2} \pm \frac{a-b}{2}$$

Taking positive value

$$= \frac{a+b}{2} + \frac{a-b}{2}$$

$$= \frac{a}{2} + \frac{b}{2} + \frac{a}{2} - \frac{b}{2}$$

$$= \frac{a}{2} + \frac{a}{2} + \frac{b}{2} - \frac{b}{2}$$

$$= \frac{a}{2} + \frac{a}{2} + 0$$

$= a$

Taking negative value

$$= \frac{a+b}{2} - \left(\frac{a-b}{2}\right)$$

$$= \frac{a}{2} + \frac{b}{2} - \frac{a}{2} + \frac{b}{2}$$

$$= \frac{a}{2} - \frac{a}{2} + \frac{b}{2} + \frac{b}{2}$$

$$= 0 + \frac{b}{2} + \frac{b}{2}$$

$= b$

Thus, $A+\sqrt{(A+G)(A-G)} = a$

and $A-\sqrt{(A+G)(A-G)} = b$

Hence, proved.

Example 2.5: The ratio of the A.M. and G.M. of two positive numbers a and b, is $m : n$. Show that $a : b = (m+\sqrt{m^2-n^2}) : (m-\sqrt{m^2-n^2})$ [NCERT]

Ans. Here, the two numbers be a and b

Arithmetic mean $=$ A.M. $= \frac{a+b}{2}$ and geometric mean $=$ G.M. $= \sqrt{ab}$

According to the question,

$$\frac{A.M.}{G.M.} = \frac{m}{n}$$

$$\frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$$

On applying componendo and dividendo, we get

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}}=\frac{m+n}{m-n}$$

$$\frac{(\sqrt{a})^2+(\sqrt{b})^2+2(\sqrt{a}\times\sqrt{b})}{(\sqrt{a})^2+(\sqrt{b})^2-2(\sqrt{a}\times\sqrt{b})}=\frac{m+n}{m-n}$$

$[\because\ (x+y)^2=x^2+y^2+2xy$
$(x-y)^2=x^2+y^2-2xy]$

$$\frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2}=\frac{m+n}{m-n}$$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^2=\frac{m+n}{m-n}$$

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\sqrt{\frac{m+n}{m-n}}$$

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{m+n}}{\sqrt{m-n}}$$

On applying componendo and dividendo, we get

$$\frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})}=\frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}$$

$$\frac{2\sqrt{a}}{2\sqrt{b}}=\frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}$$

$$\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}$$

On squaring both sides, we get

$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2=\left(\frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}\right)^2$$

$$\frac{(\sqrt{a})^2}{(\sqrt{b})^2}=\frac{(\sqrt{m+n}+\sqrt{m-n})^2}{(\sqrt{m+n}-\sqrt{m-n})^2}$$

$[\because\ (x+y)^2=x^2+y^2+2xy$
$(x-y)^2=x^2+y^2-2xy]$

$$\frac{a}{b}=\frac{(\sqrt{m+n})^2+(\sqrt{m-n})^2+2(\sqrt{m+n})(\sqrt{m-n})}{(\sqrt{m+n})^2+(\sqrt{m-n})^2-2(\sqrt{m+n})(\sqrt{m-n})}$$

$$\frac{a}{b}=\frac{m+n+m-n+2\sqrt{(m+n)(m-n)}}{m+n+m-n-2\sqrt{(m+n)(m-n)}}$$

$$\frac{a}{b}=\frac{m+m+n-n+2\sqrt{(m^2-n^2)}}{m+m+n-n-2\sqrt{(m^2-n^2)}}$$

$$\frac{a}{b}=\frac{2m+2\sqrt{(m^2-n^2)}}{2m-2\sqrt{(m^2-n^2)}}$$

$$\frac{a}{b}=\frac{2\left(m+\sqrt{(m^2-n^2)}\right)}{2\left(m-\sqrt{(m^2-n^2)}\right)}$$

$$\frac{a}{b}=\frac{m+\sqrt{(m^2-n^2)}}{m-\sqrt{(m^2-n^2)}}$$

Thus, $a:b=(m+\sqrt{m^2-n^2}):(m-\sqrt{m^2-n^2})$

Hence, proved.

Example 2.6: If a, b, c are in A.P., b, c, d are in G.P. and $\frac{1}{c},\frac{1}{d},\frac{1}{e}$ are in A.P. prove that a, c, e are in G.P.

[NCERT]

Ans. It is given that a, b, c are in A.P.

So, their common difference is same

$$b-a=c-b$$
$$b+b=c+a$$
$$2b=c+a$$
$$b=\frac{c+a}{2} \quad \text{...(i)}$$

Also, given that b,c,d are in G.P.

So, their common ratio is same

$$\frac{c}{b}=\frac{d}{c}$$
$$c^2=bd \quad \text{...(ii)}$$

Also,

$\frac{1}{c},\frac{1}{d},\frac{1}{e}$ are in A.P.

So, their common difference is same

$$\frac{1}{d}-\frac{1}{c}=\frac{1}{e}-\frac{1}{d}$$
$$\frac{1}{d}+\frac{1}{d}=\frac{1}{e}+\frac{1}{c}$$
$$2\left(\frac{1}{d}\right)=\frac{c+e}{ec}$$
$$\frac{2}{d}=\frac{c+e}{ec}$$
$$\frac{d}{2}=\frac{ec}{c+e}$$
$$d=2\left(\frac{ec}{c+e}\right) \quad \text{...(iii)}$$

We need to show that a, c, e are in G.P. i.e., we need to show their common ratio is same.

$$\therefore \quad \frac{c}{a}=\frac{e}{c}$$
$$\Rightarrow \quad c^2=ae$$

So, we need to show $c^2=ae$.

From (ii), we have

$$c^2 = bd$$

Putting value of $b = \dfrac{a+c}{2}$ and $d = \dfrac{2ce}{c+e}$

$$c^2 = \left(\frac{a+c}{2}\right)\left(\frac{2ce}{c+e}\right)$$

$$c^2 = \frac{(a+c)(2ce)}{2(c+e)}$$

$$c^2 = \frac{(a+c)(ce)}{(c+e)}$$

$$\frac{c^2}{c} = \frac{e(a+c)}{c+e}$$

$$c = \frac{e(a+c)}{c+e}$$

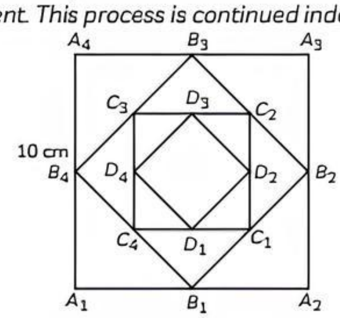
$$c(c+e) = e(a+c)$$
$$c^2 + ec = ea + ec$$
$$c^2 = ea + ec - ec$$
$$c^2 = ea + 0$$
$$c^2 = ea$$

Thus, a, c and e are in G.P.

Hence, proved

Example 2.7: Case Based:

A student of class XI draw a square of side 10 cm. Another student join the mid-point of this square to form new square. Again, the mid-points of the sides of this new square are joined to form another square by another student. This process is continued indefinitely.



Based on the above information, answer the following questions.

(A) Assertion (A): The side of fourth square is $\dfrac{5}{\sqrt{2}}$ cm.

Reason (R): GP is a type of sequence where each succeeding term is produced by adding each preceding term by a fixed number.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).
(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).
(c) (A) is true but (R) is false.
(d) (A) is false but (R) is true.

(B) The area (in sq cm) of the fifth square is:

(a) $\dfrac{25}{2}$ (b) 50
(c) 25 (d) $\dfrac{25}{4}$

(C) The perimeter (in cm) of the 7^{th} square is:

(a) 10 (b) 20
(c) 5 (d) $\dfrac{5}{2}$

(D) Find the sum of areas (in sq cm) of all the square formed.

(E) Find the sum of the perimeter (in cm) of all the square formed.

Ans. Let A_1, A_2, A_3, A_4 be the vertices of the first square with each side equal to 10 cm. Let B_1, B_2, B_3, B_4 be the mid-point of its side.

Then, $$B_1B_2 = \sqrt{A_2B_1^2 + A_2B_2^2}$$
$$= \sqrt{5^2+5^2} = \sqrt{25+25} = 5\sqrt{2}$$
$$C_1B_2 = B_2C_2 = \frac{5\sqrt{2}}{2}$$

Similarly, $$C_1C_2 = \sqrt{B_2C_2^2 + B_2C_1^2}$$
$$= \sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{2}\right)^2}$$
$$= \sqrt{\frac{25}{2}+\frac{25}{2}} = \sqrt{25} = 5 \text{ cm}$$

Similarly, the side of fourth square is $\dfrac{5}{\sqrt{2}}$ cm.

$\therefore$ Sides of each square are

$10, 5\sqrt{2}, 5, \dfrac{5}{\sqrt{2}}, \dfrac{5}{2}, \dfrac{5}{2\sqrt{2}}, \dfrac{5}{4}, \ldots$ respectively

which form a GP with $a = 10$ and $r = \dfrac{1}{\sqrt{2}}$.

(A) *(c) (A) is true but (R) is false.*

Explanation:

Side of fourth square $= ar^3 = 10\left(\dfrac{1}{\sqrt{2}}\right)^3$

$$= \frac{10}{2\sqrt{2}} = \frac{5}{\sqrt{2}} \text{ cm}$$

We know that, GP is a type of sequence where each succeeding term is produced by multiplying each preceeding term by a fixed number.

(B) (d) $\frac{25}{4}$

Explanation:

Side of fifth square = $ar^4 = 10\left(\frac{1}{\sqrt{2}}\right)^4$

$= \frac{10}{4} = \frac{5}{2}$

$\therefore$ Area of fifth square = $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ cm^2

(C) (c) 5

Explanation:

The side of 7th square = $ar^6 = 10\left(\frac{1}{\sqrt{2}}\right)^6$

$= \frac{10}{8} = \frac{5}{4}$

$\therefore$ Perimeter of 7th square = $\frac{5}{4} \times 4 = 5$ cm

(D) Sum of areas of all square formed is

$10^2 + (5\sqrt{2})^2 + (5)^2 + \left(\frac{5}{\sqrt{2}}\right)^2 ...$

$= 100 + 50 + 25 + \frac{25}{2} + ...$

Here, $a = 100$, $r = \frac{50}{100} = \frac{1}{2}$, which is an infinite GP.

$= \frac{a}{1-r} = \frac{100}{1-\frac{1}{2}} = 200$ cm^2

(E) Sum of perimeter of all square is

$4\left(10 + 5\sqrt{2} + 5 + \frac{5}{\sqrt{2}} ...\right)$

$= 4 \times \frac{10}{1-\frac{1}{\sqrt{2}}} = \frac{40\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$

$= \frac{40\sqrt{2}(\sqrt{2}+1)}{(2-1)} = 80 + 40\sqrt{2}$ cm

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. **If the arithmetic and geometric means of two numbers are 10 and 8 respectively, then one number exceeds the other number by:**

   (a) 8 (b) 10

   (c) 12 (d) 16

   [Delhi Gov. QB 2022]

Ans. (c) 12

Explanation: Let the numbers be a, b.

So, arithmetic mean = $\frac{a+b}{2}$

$\Rightarrow \frac{a+b}{2} = 10$

$\Rightarrow a + b = 20$

Geometric mean = $\sqrt{ab}$

$\Rightarrow \sqrt{ab} = 8$

$\Rightarrow ab = 64$

Since, $a + b = 20$

$\Rightarrow b = 20 - a$

$\Rightarrow a \times (20 - a) = 64$

$\Rightarrow a^2 - 20a + 64 = 0$

$\Rightarrow a^2 - 16a - 4a + 64 = 0$

$\Rightarrow (a - 16) \times (a - 4) = 0$

$\Rightarrow a = 16$ or $a = 4$.

So, the numbers are 16, 4.

Hence, the one number exceeds the other number by 16 – 4 *i.e.,* 12.

2. **The sum of n terms of a G.P. with first term a and common ratio r where $r > 1$, is:**

   (a) $a\left(\frac{1-r^n}{1-r}\right)$ (b) $a\left(\frac{1-r}{1-r^n}\right)$

   (c) $a\left(\frac{r^n-1}{r-1}\right)$ (d) None

Ans. (c) $a\left(\frac{r^n-1}{r-1}\right)$

Explanation: The formula of sum of G.P. of n term is

$S_n = a\left(\frac{r^n-1}{r-1}\right)$ if $r > 1$

$= a\left(\frac{1-r^n}{1-r}\right)$ if $r < 1$

3. If A be one A.M. and p, q be two GM's between two numbers, then $2A$ is equal to:

(a) $\frac{p^3+q^3}{pq}$ (b) $\frac{p^3-q^3}{pq}$

(c) $\frac{p^2+q^2}{2}$ (d) $\frac{pq}{2}$

[Delhi Gov. QB 2022]

Ans. *(a)* $\frac{p^3+q^3}{pq}$

Explanation: Let the two positive numbers be a and b

a, A and b are in A.P.

$\therefore \quad 2A = a + b$ ...(i)

Also, a, p, q and b are in G.P.

$\therefore \quad r = \left(\frac{b}{a}\right)^{1/3}$

Again, $p = ar$ and $q = ar^2$ ...(ii)

Now, $2A = a + b$ [From (i)]

$= a + a\left(\frac{b}{a}\right)$

$= a + a\left(\left(\frac{b}{a}\right)^{\frac{1}{3}}\right)^3$

$= a + ar^3$

$= \frac{(ar)^2}{ar^2} + \frac{(ar^2)^2}{ar}$

$= \frac{p^2}{q} + \frac{q^2}{p}$ [Using (ii)]

$= \frac{p^3+q^3}{pq}$

4. $3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \times ...$ upto infinite terms is:

(a) 3^2 (b) 3

(c) 3^3 (d) 3^4

Ans. *(b)* 3

Explanation: Given

$3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \times ...$

$= 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...}$

$= 3^{\frac{1}{2}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+...\right)}$

$= 3^{\frac{1}{2}\times\left(\frac{1}{1-\frac{1}{2}}\right)}$ $[\because a = 1 \text{ and } r = \frac{1}{2}]$

$= 3^{\frac{1}{2}\times\frac{2}{1}} = 3$

5. The geometric mean of 2 and 8 is:

(a) 4 (b) 6

(c) 7 (d) 5

Ans. *(a)* 4

Explanation: The geometric mean of 2 and 8 is $\sqrt{16}$ *i.e.*, 4.

6. Four geometric means between 3 and 96 are:

(a) 6, 12, 24, 48 (b) 6, 10, 24, 48

(c) 6, 10, 40, 48 (d) 48, 24, 10, 5

Ans. *(a)* 6, 12, 24, 48

Explanation: Let G_1, G_2, G_3 and G_4 be the required GM's.

Then, 3, G_1, G_2, G_3, G_4, 96 are in G.P.

Let r be the common ratio. Here, 96 is the 6^{th} term.

$\therefore \quad 96 = ar^{6-1} = 3r^5$

$\Rightarrow \quad 32 = r^5$

$\Rightarrow \quad (2)^5 = r^5$

$\Rightarrow \quad r = 2$

$\therefore \quad G_1 = ar = 3 \times 2 = 6$

$G_2 = ar^2 = 3 \times 2^2 = 12$

$G_3 = ar^3 = 3 \times 2^3 = 24$

And $G_4 = ar^4 = 3 \times 2^4 = 48$

7. The minimum value of $4^x + 4^{1-x}$, $x \in R$ is:

(a) 2 (b) 4

(c) 1 (d) 0 [NCERT Exemplar]

Ans. *(b)* 4

Explanation: We know that, $AM \geq GM$

$\Rightarrow \frac{4^x+4^{1-x}}{2} \geq \sqrt{4^x . 4^{1-x}}$

$\Rightarrow 4^x + 4^{1-x} \geq 2.2$

$\Rightarrow 4^x + 4^{1-x} \geq 4$

Assertion Reason Questions

Direction: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

8. Assertion (A): The sum of first n terms of the series 0.6 + 0.66 + 0.666 + ... is $\frac{2}{3}\left[n-\frac{1}{9}\left\{1-\left(\frac{1}{10}\right)^n\right\}\right]$.

Reason (R): General term of a GP is $T_n = ar^{n-1}$, where a = first term and r = common ratio.

Ans. (b) *Both (A) and (R) are true but (R) is not the correct explanation of (A).*

Explanation: Let

S = 0.6 + 0.66 + 0.666 + ... upto n terms

S = 6(0.1 + 0.11 + 0.111 + ... upto n terms)

$$= \frac{6}{9}(0.9+0.99+0.999+\ldots \text{ upto } n \text{ terms})$$

$$= \frac{2}{3}\left[\frac{9}{10}+\frac{99}{100}+\frac{999}{1000}+\ldots\text{upto } n \text{ terms}\right]$$

$$= \frac{2}{3}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right) +\ldots \text{ upto } n \text{ terms}\right]$$

$$= \frac{2}{3}\left[(1+1+1+\ldots\text{ upto } n \text{ terms})-\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots\text{upto } n \text{ terms}\right)\right]$$

$$= \frac{2}{3}\left[n-\frac{\frac{1}{10}\left\{1-\left(\frac{1}{10}\right)^n\right\}}{1-\frac{1}{10}}\right]$$

$$\left[\because \text{ sum of } GP = \frac{a(1-r^n)}{1-r}, r<1\right]$$

$$= \frac{2}{3}\left[n-\frac{\frac{1}{10}\left\{1-\left(\frac{1}{10}\right)^n\right\}}{\frac{9}{10}}\right]$$

$$= \frac{2}{3}\left[n-\frac{1}{9}\left\{1-\left(\frac{1}{10}\right)^n\right\}\right]$$

9. Assertion (A): The sum of first n natural number is 1 + 2 + 3 + ... + n $= n\left(\frac{n+1}{2}\right)$.

Reason (R): For n number of A.P. the sum $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$.

Ans. (a) *Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: We know that the sum of first n natural number is $n\left(\frac{n+1}{2}\right)$

For A.P.

1 + 2 + ... n

$a = 1, d = 1$

$$S_n = \frac{n}{2}[2\times1+(n-1)1]$$

$$= \frac{n}{2}[2+n-1]$$

$$= \frac{n}{2}[n+1]$$

Hence, both assertion and reason are true and reason is the correct explanation of assertion.

10. Assertion (A): Two sequence can be in both A.P. and G.P.

Reason (R): If the sum of n terms of a sequence is a quadratic expression, then it always represents an A.P.

Ans. (c) *(A) is true but (R) is false.*

Explanation: Two sequence can be in both A.P. and G.P. in some special cases.

If the sum of n term of a sequence is quadratic then it may or may not be represent an A.P.

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

11. *A person was employed in a company on the condition that he will be paid rupees 2, 4, 6, 8, 10, ... for the work done in the company on 1st, 2nd, 3rd, ... day.*

(A) Name the progression of amount received by the person.

(a) A.P. (b) G.P.

(c) A.P. and G.P. (d) None

(B) The common difference of the sequence is:

(a) 1 (b) 2

(c) 3 (d) 4

(C) The first term of the sequence is:

(a) 2 (b) 4

(c) 6 (d) 8

(D) The total amount received in 10 days in rupees is:

(a) 115 (b) 105

(c) 120 (d) 110

(E) Assertion (A): If the numbers $\frac{-2}{7}$, k, $\frac{-7}{2}$ are in GP, then $k = \pm 1$.

Reason (R): If a_1, a_2, a_3 are in GP, then $\frac{a_2}{a_1} = \frac{a_3}{a_2}$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

Ans. (A) *(a) A.P.*

Explanation: The given series 2, 4, 6, 8, ... is A.P. because the common difference is same as

$$a_2 - a_1 = 2$$
$$a_3 - a_2 = 2$$

(B) *(b) 2*

Explanation: The common difference as the A.P. 2, 4, 6, 8,...

$$a_2 - a_1 = 4 - 2 = 2$$
$$a_3 - a_2 = 6 - 4 = 2$$
$$a_4 - a_3 = 8 - 6 = 2$$

(C) *(a) 2*

Explanation: The first term of A.P 2, 4, 6, 8 ,... is 2.

(D) *(d) 110*

Explanation: Here $a = 2$, $d = 2$ and $n = 10$

So,

$$S_{10} = \frac{10}{2}[2 \times 2 + (10-1)2]$$
$$= \frac{10}{2}[4 + 18]$$
$$= \frac{10}{2}[22]$$
$$= 10 \times 11 = ₹\,110$$

(E) *(a) Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: If $-\frac{2}{7}, k, -\frac{7}{2}$ are in *GP*

Then, $\frac{a_2}{a_1} = \frac{a_3}{a_2}$

$$\left[\because \text{common ration } (r) = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots\right]$$

$$\frac{k}{-\frac{2}{7}} = \frac{-\frac{7}{2}}{k}$$

$$\Rightarrow \frac{7}{-2}k = \frac{-7}{2} \times \frac{1}{k}$$

$$\Rightarrow 2k \times 2k = -7 \times (-2)$$

$$\Rightarrow 14k^2 = 14$$

$$\Rightarrow k^2 = 1$$

$$\Rightarrow k = \pm 1$$

12. *A group of students obtain the sum of first n natural number is $\frac{n(n+1)}{2}$ named as A_n, sum of square of n natural number is $\frac{n(n+1)(2n+1)}{6}$ named as B_n and sum of cubes of n natural number is $\left(\frac{n(n+1)}{2}\right)^3$ named as C_n.*

(A) Find the value of A_5 and B_3.

(B) Find the value of C_2 and the sum of first n natural number is also the sum of n numbers of A.P whose first term is 1 and common difference is also 1.

(C) What is the geometric mean of 6 and 24?

Ans. (A) $A_n = \frac{n(n+1)}{2}$

So, Put $n = 5$

We get

$$A_5 = \frac{5(5+1)}{2}$$
$$= \frac{5 \times 6}{2}$$
$$= 3 \times 5$$
$$= 15$$

$$B_n = \frac{n(n+1)(2n+1)}{6}$$

So put $n = 3$

$$B_3 = \frac{3(3+1)(6+1)}{6}$$
$$= \frac{3 \times 4 \times 7}{6}$$
$$= \frac{28}{2}$$
$$= 14$$

(B) $C_n = \left(\frac{n(n+1)}{2}\right)^3$

Put $n = 2$

$C_2 = \left(\frac{2(2+1)}{2}\right)^3$

$= (3)^3$

$= 27$

$A_n = \frac{n(n+1)}{2}$

and $S_n = 1 + 2 + 3 + \ldots n$

$= \frac{n}{2}[2a+(n-1)d]$

$= \frac{n}{2}[2+(n-1)1]$

$= \frac{n}{2}[2+n-1]$

$= \frac{n(n+1)}{2}$

(C) Given two data values are 6 and 24.

So, $n = 2$.

So, to find the geometric mean of 6 and 24, we have to take the square root for the product of 6 and 24.

Geometric Mean = $\sqrt{(6\times 24)}$

Geometric Mean = $\sqrt{(144)}$

Thus, the square root of 144 is 12.

Therefore, the geometric mean of 6 and 24 is 12.

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

13. Find the sum of infinity of the G.P $5, \frac{20}{7}, \frac{80}{49}, \ldots$.

Ans. Given G.P. is $5, \frac{20}{7}, \frac{80}{49} \ldots$

Since, $\frac{\left(\frac{20}{7}\right)}{5} = \frac{4}{7} \cdot \frac{\left(\frac{80}{49}\right)}{\left(\frac{20}{7}\right)} = \frac{4}{7}$ Clearly, $|r| < 1$.

Also, the first term of G.P. is $a = 5$.

Hence, the sum of infinity of G.P. is

$$S_\infty = \frac{a}{1-r} = \frac{5}{1-\frac{4}{7}} = \frac{35}{3}$$

14. Prove that $9^{\frac{1}{2}} \times 9^{\frac{1}{4}} \times 9^{\frac{1}{8}} \ldots \infty = 9$

Ans. Here

$= 9^{\frac{1}{2}} \times 9^{\frac{1}{4}} \times 9^{\frac{1}{8}} \ldots \infty$

$= 9^{\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\ldots\infty\right]}$

$= 9^{\left[\frac{\frac{1}{2}}{1-\frac{1}{2}}\right]}$ $\quad \because \frac{1}{2}+\frac{1}{4}+\frac{1}{8}\ldots\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

$= 9^1$

$= 9$

15. The first term of a G.P is 4 and the sum of infinity is 8 find common ratio.

Ans. Given, $S_\infty = 8$

$\Rightarrow \frac{a}{1-r} = 8$

$\Rightarrow \frac{4}{1-r} = 8$

$\Rightarrow 4 = 8 - 8r$

$\Rightarrow -4 = -8r$

Or, $\frac{1}{2} = r$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

16. If x, y, z are distinct positive number, then prove that $(x + y)(y + z)(z + x) > 8xyz$.

[NCERT Exemplar]

Ans. Using A.M > G.M, we obtain

$\frac{x+y}{2} > \sqrt{xy}, \frac{y+z}{2} > \sqrt{yz}$ and $\frac{z+x}{2} > \sqrt{zx}$ for the three pairs

$\Rightarrow \quad x+y>2\sqrt{xy}, y+z>2\sqrt{yz}$

and $z+x>2\sqrt{zx}$

$\Rightarrow \quad (x+y)(y+z)(z+x)$

$>2\sqrt{xy}\times 2\sqrt{xy}\times, 2\sqrt{zx}$

$\Rightarrow \quad (x+y)(y+z)(z+x) > 8xyz.$

17. **If $x\in R$, find the minimum value of the expression $3^x + 3^{1-x}$.** **[NCERT Exemplar]**

Ans. We know that, A.M. > G.M.

$$\therefore \quad \frac{3^x+3^{1-x}}{2} \geq \sqrt{3^x \times 3^{1-x}} \text{ for all } x\in R$$

$$\Rightarrow \quad \frac{3^x+3^{1-x}}{2} \geq \sqrt{3} \text{ for all } x\in R$$

$$\Rightarrow \quad 3^x+3^{1-x} \geq 2\sqrt{3} \text{ for all } x\in R$$

Hence, the minimum value of $3^x + 3^{1-x}$ for any $x\in R$ is $2\sqrt{3}$.

18. **Write the sum of 2 + 4 + 6 + 8 + ... + 2n.**

Ans. Here, n^{th} term is $2n$

So, $\quad S_n = \sum 2n$

$= 2\sum n$

$= \frac{2n(n+1)}{2}$

$= n(n+1)$

$= n^2 + n$

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

19. **Find the sum of the sequence 2, 22, 222, 2222, ...**

Ans. Let S_n = 2 + 22 + 222 + 2222 + ...

$= 2(1 + 11 + 111 + ...)$

$= \frac{2}{9}(9+99+999+...)$

$= \frac{2}{9}((10-1)+(10^2-1)+(10^3-1)+...)$

$= \frac{2}{9}(10+10^2+10^3+...-n)$

Since, 10, 100, 1000 .... n terms are in G.P. with a = 10 and $r = \frac{100}{10} = 10$

$$\therefore \quad S_n = \frac{2}{9}\left[10\left(\frac{10^n-1}{10-1}\right)-n\right]$$

$$= \frac{2}{9}\left[\frac{10}{9}(10^n-1)-n\right]$$

$$= \frac{2}{81}[10(10^n-1)-9n$$

20. **Find the positive numbers whose difference is 32 and whose A.M exceeds the G.M. by 8.**

Ans. Let the two numbers be a and b

Then, as per the question, $a - b = 32$

It is also given that

A.M. – G.M = 8

$$\frac{a+b}{2} - \sqrt{ab} = 8$$

Or, $\quad a+b-2\sqrt{ab} = 16$

Or, $\quad (\sqrt{a}-\sqrt{b})^2 = 16$

Or, $\quad \sqrt{a}-\sqrt{b} = \pm 4 \quad$ ...(i)

Now, $\quad a - b = 32$

$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = 32$

$(\sqrt{a}+\sqrt{b})\times(\pm 4) = 32$

$\sqrt{a}+\sqrt{b} = \pm 8 \quad$ ...(ii)

On solving (i) and (ii), we get

$a = 36$

$b = 4$

21. **Find the sum of first n terms of the series**

$$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+...+n \text{ terms}$$

Ans. The given sequence is

$$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+...+n \text{ terms}$$

We can write each individuals as,

$$\frac{1}{2} = 1-\frac{1}{2}$$

$$\frac{3}{4} = 1-\frac{1}{4}$$

$$\frac{7}{8} = 1-\frac{1}{8} \text{ and ... till n terms}$$

Now writing each term in its new form, we get,

$$\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{8}\right)+...+\left(1-\frac{1}{2^n}\right)$$

$$= (1+1+1 \times \_\ n \text{ terms}) - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\_+\frac{1}{2^n}\right)$$

$$= n - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\_+\frac{1}{2^n}\right) \quad \text{...(i)}$$

Now, $\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\_+\frac{1}{2^n}\right)$ in a G.P with common ratio = $\frac{1}{2}$ so,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\_+\frac{1}{2^n} = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$$

$$= \left[1-\left(\frac{1}{2}\right)^n\right]$$

Putting this value in eq. (i), we get

$$n - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\_+\frac{1}{2^n}\right)$$

$$= n - \left[1-\left(\frac{1}{2}\right)^n\right] = n-1+\left(\frac{1}{2}\right)^n$$

22. Write first five terms of sequence of whose n^{th} term is $\frac{1}{2}|2n-1|$.

Ans. Here, the n^{th} term of sequence is given by $\frac{1}{2}|2n-1|$

On putting $n = 1$, we get

$$a_1 = \frac{1}{2}|2\times1-1| = \frac{1}{2}$$

$$\therefore \quad a_2 = \frac{1}{2}|2\times2-1| = \frac{3}{2}$$

$$\therefore \quad a_3 = \frac{1}{2}|2\times3-1| = \frac{5}{2}$$

$$\therefore \quad a_4 = \frac{1}{2}|2\times4-1| = \frac{7}{2}$$

$$\therefore \quad a_5 = \frac{1}{2}|2\times5-1| = \frac{9}{2}$$

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

23. If A is the arithmetic mean and G_1, G_2 be two geometric means between any two numbers, then prove that

$$2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} \qquad \text{[Diksha]}$$

Ans. Let the two numbers be x and y

$$\therefore \quad A = \frac{x+y}{2} \quad \text{...(i)}$$

If G_1 and G_2 be the geometric means between x and y then x, G_1, G_2, y are in GP.

Then $\quad y = xr^{4-1} \quad [\because a_n = ar_{n-1}]$

$\Rightarrow \quad y = xr^3$

$\Rightarrow \quad \frac{y}{x} = r^3$

$\Rightarrow \quad r = \left(\frac{y}{x}\right)^{\frac{1}{3}}$

Now $\quad G_1 = xr$

$$= x\left(\frac{y}{x}\right)^{\frac{1}{3}} \qquad \left[\because r = \left(\frac{y}{x}\right)^{\frac{1}{3}}\right]$$

and $\quad G_2 = xr^2$

$$= x\left(\frac{y}{x}\right)^{\frac{2}{3}}$$

$\therefore$ From R.H.S.

$$\frac{G_1^2}{G_2}+\frac{G_2^2}{G_1} = \frac{x^2\left(\frac{y}{x}\right)^{\frac{2}{3}}}{x\left(\frac{y}{x}\right)^{\frac{2}{3}}} + \frac{x^2\left(\frac{y}{x}\right)}{x\left(\frac{y}{x}\right)^{\frac{1}{3}}}$$

$$= x + x\left(\frac{y}{x}\right)^{\frac{4}{3}-\frac{1}{3}}$$

$= x + x\left(\frac{y}{x}\right)$

$= x + y = 2A$

[Using equation(i)]

$\therefore$ LHS = RHS

24. (A) If a, b, c, d are four distinct positive quantities in A.P., then show that $bc > ad$.

(B) If a, b, c, d are four distinct positive quantities in G.P., then show that $a + b > b + c$.

Ans. (A) Given that a, b, c, d are four distinct positive quantities in A.P.

Since, a, b, c are in A.P.

So, $\frac{a+c}{2} = b$

Now, A.M. > G.M. [$\because$ a, c are distinct]

$\Rightarrow \frac{a+c}{2} > \sqrt{ac}$

$\Rightarrow b > \sqrt{ac}$

$\Rightarrow b^2 > ac$ [On squaring both sides] ...(i)

Since, b, c, d are in A.P.

So, $\frac{b+d}{2} = c$

Now, A.M. > G.M. [$\because$ b, d are distinct]

$\Rightarrow \frac{b+d}{2} > \sqrt{bd}$

$\Rightarrow c > \sqrt{bd}$

$\Rightarrow c^2 > bd$ ...(ii)

Multiplying (i) and (ii), we get

$b^2 c^2 > (ac)(bd)$

$\Rightarrow bc > ad$

(B) Given a, b, c, d are four distinct positive quantities in G.P.

Since a, b, c are in G.P.

So, $\sqrt{ac} = b$

Now, A.M. > G.M. [$\because$ a, c are distinct]

$\Rightarrow \frac{a+b}{2} > \sqrt{ac}$

$\Rightarrow \frac{a+b}{2} > b$

$\Rightarrow a + b > 2b$ ...(iii)

Since b, c, d are in G.P.

So, $\sqrt{bd} = c$

Now, A.M. > G.M. [$\because$ a, c are distinct]

$\Rightarrow \frac{b+d}{2} > \sqrt{bd}$

$\Rightarrow \frac{b+d}{2} > c$

$\Rightarrow b + d > 2c$ ...(iv)

On adding (iii) and (iv), we get

$(a + c) + (b + d) > 2b + 2c$

$\Rightarrow a + d > b + c$.

Hence, proved

25. If p^{th}, q^{th} and r^{th} terms of an A.P. and G.P. are both a, b and c respectively, show that $a^{b-c} . b^{c-a} . c^{a-b} = 1$ **[Diksha]**

Ans. Let A be the first term and D be the common difference of the AP. Therefore

$a_p = A + (p - 1)D = a$ ...(i)

$a_q = A + (q - 1)D = b$ ...(ii)

$a_r = A + (r - 1)D = c$ ...(iii)

Also suppose A' be the first term and R be the common ratio of the G.P. Therefore,

$a_p = A'R^{p-1} = a$ ...(iv)

$a_q = A'R^{q-1} = b$ ...(v)

$a_r = A'R^{r-1} = c$ ...(vi)

Now,

Subtracting (ii) from (i), we get

$A + (p - 1)D - A - (q - 1)D = a - b$

$\Rightarrow (p - q)D = a - b$ ...(vii)

Subtracting (iii) from (ii), we get

$A + (q - 1)D - A - (r - 1)D = b - c$

$\Rightarrow (q - r)D = b - c$ ...(viii)

Subtracting (i) from (iii), we get

$A + (r - 1)D - A - (p - 1)D = c - a$

$\Rightarrow (r - p)D = c - a$ ...(ix)

$\therefore a^{b-c} b^{c-a} c^{a-b}$

$= [A'R^{(p-1)}]^{(q-r)D} \times [A'R^{(q-1)}]^{(r-p)D} \times [A'R^{(r-1)}]^{(p-q)D}$

[Using eq. (iv), (v), (vi), (vii), (viii) and (ix)]

$= [A'R^{[(q-r)D+(r-p)D+(p-q)D](q-r)D]}$

$\times R^{[(p-1)(q-r)D+(q-q)(r-p)D+(r-1)(p-q)D]}$

$= A'^{[q-r+r-p+p-q]D}$

$\times R^{[pq-pr-q+r+qr-pq-r+p+pr-qr-p+q]D}$

$= (A')^0 \times R^0$

$= 1 \times 1$

$= 1$